Here are brief notes about topics covered in class on complex numbers that are not covered in the textbook.

- $i$ as the solution to $x^2 + 1 = 0$, that is, $i = \sqrt{-1}$.
- Complex numbers: $z = x + iy$, with $x$ and $y$ real numbers, called the real and imaginary parts of $z$, $x = \text{Re}(z)$ and $y = \text{Im}(z)$.
- Square roots of other negative numbers.
- Solution to any quadratic equation.
- Fundamental theorem of algebra: Every polynomial has at least one root. This easily implies by induction that every polynomial can be factorized as $a_n(z - w_1)(z - w_2) \cdots (z - w_n)$. For a proof of this see below.
- Complex conjugate: $z^* = x - iy$.
- Inverse of a complex number:
  \[
  \frac{1}{z} = \frac{z^*}{z^* z} = \frac{x - iy}{x^2 + y^2}.
  \]
- Modulus $|z| = \sqrt{z^* z} = \sqrt{x^2 + y^2}$.
- The complex plane: representation of complex numbers as points or vectors on a plane, with $\text{Re}(z)$ on the horizontal axis and $\text{Im}(z)$ on the vertical axis. The distance from the origin to $z$ is $|z|$.
- Euler’s identity: $e^{i\theta} = \cos \theta + i \sin \theta$. We proved this two ways: (a) Expand both sides in a power series and show they are equal term by term, and (b) note that both sides are equal to their own derivative with respect to $\theta$, so they satisfy the same first order differential equation. They are also obviously equal when $\theta = 0$, so they are equal everywhere.
- $e^{i\pi} + 1 = 0$, a remarkable equation, involving 0, 1, $e$, $\pi$ and $i$. 

• Polar form of a complex number $z = re^{i\theta}$, where $r$ and $\theta$ are the polar coordinates of $z$ on the complex plane, $r = |z|$ and $\theta = \tan^{-1}(y/x)$. The angle $\theta$ is called the angle or argument of $z$, $\text{Arg}(z)$. There are two arctangents within the interval $\theta \in (-\pi, \pi)$; the correct one has the same sign as $y$. Conversely, $x = r \cos \theta$ and $y = r \sin \theta$.

• $z^{1/n} = (re^{i\theta + i2\pi m})^{1/n} = r^{1/n}e^{i\theta/n + i2\pi m/n}$. These are distinct for $m = 0, 1, \ldots, n - 1$, so there are $n$ $n$th roots of $z$. Given any one, you can get the others by multiplying by the $n$th roots of unity, $e^{i2\pi m/n}$, $m = 0, 1, \ldots, n - 1$. Note that if $w$ is an $n$th root of unity, then so is $w^*$. In class the example of $(8i)^{1/3}$ was worked out in detail in polar form, Cartesian form, and graphical form on the complex plane. We found that in cartesian form the roots are $\sqrt{3} \pm i$ and $-2i$.

• Computing moduli: $|e^{i\theta}| = 1$ for all real $\theta$, and $|z^*| = |z|$.

• Geometric interpretations of conjugation and multiplication: $z \rightarrow z^*$ is reflection across the real axis, and $z \rightarrow wz$ is scaling by $|w|$ and rotation counterclockwise by $\text{Arg}(w)$. In other words, when you multiply two complex numbers, the moduli multiply and the angles add.

• Note that multiplication by $i$ is counterclockwise 90 degree rotation. Thus $i^2$ is counterclockwise 180 degree rotation…which explains why $i^2 = -1$ from a geometrical perspective!

• What is $e^x$? Look for a function $f(x)$ with the property that $df/dx = f$. We try a series: $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$. Imposing equality of $df/dx$ and $f$ term by term we discover $a_n = a_0/n!$, so $f(x) = a_0 e^x$, with $e^x = \sum_{n=0}^{\infty} x^n/n!$.

• Proof of $e^{a+b} = e^a e^b$. Can expand each of the three series and show by further expanding $(a+b)^n$ that the two sides of the equation are equal. Another approach: define $f(a) = e^a e^b$ and $g(a) = e^{a+b}$, and note that $df/da = f$ and $dg/da = g$. Also $f(0) = g(0)$. So $f$ and $g$ are equal at one point and satisfy the same first order differential equation, so they are equal everywhere.

• The fundamental theorem of algebra can be proved as follows. Let $p(z)$ be an $n$th order polynomial with nonzero constant term, i.e. $p(0) \neq 0$. Then $p(re^{i\theta})$ is a closed curve in the complex plane for each $r$. When $r = 0$ this curve is just the point $p(0)$. For sufficiently large $r$ it is
dominated by the highest order term in the polynomial, \(a_nz^n\), so it is approximately \(a_n r^n e^{i\theta}\), which follows a circle of radius \(|a_n| r^n\) and loops around \(n\) times. As \(r\) grows, the \(r = 0\) curve deforms into this large \(n\)-fold circle that encloses the origin. Somewhere along the way the curve must have crossed \(z = 0\) at least once, hence there is at least one root.

- **Analytic functions**: A complex function \(h(z)\) is analytic if its derivative \(dh/dz\) is independent of the direction \(dz\) takes in the complex plane. The prototypical non-analytic function is \(h(z) = z^* = x - iy\). Why? Well the derivative in the \(x\) direction is 1, while the derivative in the \(iy\) direction is \(d(-iy)/d(iy) = -1\), so the two have opposite signs. More generally, if \(dz = d(|z| e^{i\varphi}) = e^{i\varphi} d|z|\), with \(\varphi\) fixed, then \(dz^*/dz = e^{-i2\varphi}\), which depends on direction. Hence the function is not analytic.

- **Cauchy-Riemann equations**: The text expressed these in terms of the real and imaginary parts \(f\) and \(g\) of \(h(z) = f(z) + ig(z)\), but it’s nicer not to break up \(h\) this way. To rederive the relation: the derivative \(dh/dz\) with \(dz = dx\) is \(\partial_x h\), while with \(dz = d(iy) = idy\) it is \(-i\partial_y h\) (since \(1/i = -i\)). Analyticity implies these two are equal, so

\[
\partial_y h = i\partial_x h \quad \text{complex form of Cauchy-Riemann equation (1)}
\]

Using the Cauchy-Riemann equation we can see that an analytic function satisfies a number of amazing properties.

- If \(h\) is analytic then it is harmonic, i.e. it satisfies Laplace’s equation:

\[
\nabla^2 h = \partial_x^2 h + \partial_y^2 h \quad (2)
\]

\[
= \partial_x^2 h + i^2 \partial_x^2 h. \quad (3)
\]

\[
= 0. \quad (4)
\]

This complex equation implies that the real and imaginary parts \(f\) and \(g\) are separately both harmonic.

- If \(h\) is analytic then its gradient has zero norm:

\[
\nabla h \cdot \nabla h = (\partial_x h)^2 + (\partial_y h)^2 \quad (5)
\]

\[
= (\partial_x h)^2 + i^2 (\partial_x h)^2 \quad (6)
\]

\[
= 0. \quad (7)
\]

The real and imaginary parts of this complex equation imply that the gradients of \(f\) and \(g\) have the same norm and are orthogonal: \(|\nabla f| = |\nabla g|\) and \(\nabla f \cdot \nabla g = 0\).
• Contour integration: $\int_C h(z) \, dz$, where $C$ is a contour (curve) in the complex plane. This is defined in the usual way an integral is defined, as a limit of sums. It can also be expressed as a line integral of the complex vector $H = h\hat{x} + ih\hat{y}$:

\[
\int h \, dz = \int h(dx + idy) \quad (8)
\]

\[
= \int (hdx + ihdy) \quad (9)
\]

\[
= \int H \cdot dr. \quad (10)
\]

• If $h$ is analytic the vector field $H$ introduced in the previous item has vanishing curl and divergence, as a consequence of the Cauchy-Riemann equation: Since we are living on the $x$-$y$ plane here, $(\nabla \times H)$ has only one component: $\partial_x H_y - \partial_y H_x = \partial_x (ih) - \partial_y h = 0$ by the complex Cauchy-Riemann equation. Similarly $\nabla \cdot H = \partial_x H_x + \partial_y H_y = \partial_x h + \partial_y (ih) = 0$.

• The integral of an analytic function around a closed contour vanishes. This follows from writing the contour integral as the line integral of $H$ and converting it to the surface integral of $\nabla \times H$ using Stokes’ theorem, and then invoking the fact that for analytic $h$ the vector field $H$ has vanishing curl.

• To get a nonzero value for the integral of a function of $h(z)$ around a closed contour the function must be singular (infinite) at some $z_0$, since otherwise it would be analytic and the above result would apply. A general class of functions to consider is ones that have an expansion $h(z) = \sum_n a_n (z - z_0)^n$ in negative and positive powers of $(z - z_0)$. If $a_n$ is non-zero for any negative $n$ then $h(z)$ is singular at $z_0$. If the number of such nonzero $a_n$ is finite the singularity is called a pole of order $|m|$, where $m$ is the highest negative power of $(z - z_0)$. A pole of order 1 is called a simple pole. The coefficient $a_{-1}$ of the $(z - z_0)^{-1}$ term in the expansion is called the residue $\text{Res} h(z_0)$ of $h(z)$ at the pole $z_0$. The integral of $h(z)$ counterclockwise around a closed contour enclosing poles at points $z_i$ can be shown to be given by

\[
\oint h(z) \, dz = 2\pi i \sum_i \text{Res} h(z_i).
\]

This is the residue theorem.
How can residues be conveniently computed? I’ll discuss a few example situations. The simplest case occurs if \( h(z) = g(z)/(z - z_0) \) with \( g(z) \) analytic and nonzero at \( z_0 \). Then \( h(z) \) has a simple pole at \( z_0 \) and the residue is just \( g(z_0) \). For example, the residue of \( e^z/(z + 2) \) at \( z = -2 \) is \( e^{-2} \). In terms of \( h(z) \), we can write this as \( \text{Res} \ h(z_0) = \lim_{z \to z_0} (z - z_0) h(z) \).

A slightly more involved case occurs if instead \( h(z) = g(z)/f(z) \) has a simple pole at \( z_0 \), where again \( g(z) \) is analytic and nonzero at \( z_0 \). Then, since \( f(z) = f'(z_0)(z - z_0) + O((z - z_0)^2) \), the residue is \( g(z_0)/f'(z_0) \). This is often a convenient, applicable rule. For example, the residue of \( (z^3 + 1)^{-1} \) at one of its three simple poles (which occur at the cube roots of \(-1\)) is \( 1/(3z^2) \) evaluated at the pole. For instance, the residue at \( z = e^{i\pi/3} \) is \( 3e^{i\pi/3} \).

A more complicated situation arises if \( h(z) = g(z)/(z - z_0)^n \) has a pole of order \( n \) at \( z_0 \), where once again I assume \( g(z) \) is analytic and nonzero at \( z_0 \). Then the residue is the coefficient of the \((n - 1)\)st term in the Taylor series of \( g(z) \) about \( z_0 \), \( g^{(n-1)}(z_0)/(n-1)! \). For example, \( z^3/(z - 3)^3 \) has a pole of order 3 at \( z = 3 \), where the residue is \( (1/2)(z^3)^{n-1} = 3z \) evaluated at \( z = 3 \), i.e. 9. In terms of \( h(z) \), this residue can be written

\[
\text{Res} \ h(z_0) = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[ (z - z_0)^n h(z) \right].
\]

In this form the result applies even if \( h(z) \) is not presented in the form \( g(z)/(z - z_0)^n \). For example consider \( (\sin z)/z^4 \). The numerator is analytic at \( z = 0 \), but it vanishes there, so this function does not have a pole of order 4 as it might at first seem to. Actually it has a pole of order 3 at \( z = 0 \),

\[
\frac{\sin z}{z^4} = \frac{z - z^3/3! + z^5/5! - \cdots}{z^4} = \frac{1}{z^3} - \frac{1}{3!} + \frac{1}{5!} z + \cdots
\]

The residue at \( z = 0 \) is \(-1/3!\). Here I found the residue by inspecting the expansion, of the numerator. It is equivalent to using the above formula, but use of the formula without making the series expansion of the sin would be more complicated, since you’d have to take the third derivative of \( (\sin z)/z \) and then evaluate it at \( z = 0 \).