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Introduction

The topic of this book is the application of mathematics to physical problems. Mathematics and physics are often taught separately. Despite the fact that education in physics relies on mathematics, it turns out that students consider mathematics to be disjoint from physics. Although this point of view may strictly be correct, it reflects an erroneous opinion when it concerns an education in the sciences. The reason for this is that mathematics is the only language at our disposal for quantifying physical processes. One cannot learn a language by just studying a textbook. In order to truly learn how to use a language one has to go abroad and start using that language. By the same token one cannot learn how to use mathematics in the physical sciences by just studying textbooks or attending lectures; the only way to achieve this is to venture into the unknown and apply mathematics to physical problems.

It is the goal of this book to do exactly that; problems are presented in order to apply mathematical techniques and knowledge to physical concepts. These examples are not presented as well-developed theory, instead, they are presented as a number of problems that elucidate the issues that are at stake. In this sense this book offers a guided tour; material for learning is presented but true learning will only take place by active exploration. In this process, the interplay of mathematics and physics is essential; mathematics is the natural language for physics while physical insight allows for a better understanding of the mathematics that is presented.

How can you use this book most efficiently?

Since this book is written as a set of problems you may frequently want to consult other material as well to refresh or deepen your understanding of material. In many places we refer to the book of Beas [19]. In addition, the books of Butkov [24], Riley et al. [87] and Arfken [5] on mathematical physics are excellent.
Introduction

In addition to books, colleagues in either the same field or other fields can be a great source of knowledge and understanding. Therefore, do not hesitate to work together with others on these problems if you are in the fortunate position to do so. This may not only make the work more enjoyable, it may also help you in getting "unstuck" at difficult moments and the different viewpoints of others may help to deepen yours.

For who is this book written?

This book is set up with the goal of obtaining a good working knowledge of mathematical physics that is needed for students in physics or geophysics. A certain basic knowledge of calculus and linear algebra is required to digest the material presented here. For this reason, this book is meant for upper-level undergraduate students or lower-level graduate students, depending on the background and skill of the student. In addition, teachers can use this book as a source of examples and illustrations to enrich their courses.

This book is evolving

This book will be improved regularly by adding new material, correcting errors and making the text clearer. The feedback of both teachers and students who use this material is vital in improving this text, please send your remarks to:

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2

Dimensional analysis

The material of this chapter is usually not covered in a book on mathematics. The field of mathematics deals with numbers and numerical relationships. It does not matter what these numbers are; they may account for physical properties of a system, but they may equally well be numbers that are not related to anything physical. Consider the expression $g = df/dt$. From a mathematical point of view these functions can be anything, as long as $g$ is the derivative of $f$. The situation is different in physics. When $f(t)$ is the position of a particle, and $t$ denotes time, then $g(t)$ is a velocity. This relation fixes the physical dimension of $g(t)$. In mathematical physics, the physical dimension of variables imposes constraints on the relation between these variables. In this chapter we explore these constraints. In Section 2.2 we show that this provides a powerful technique for spotting errors in equations. In the remainder of this chapter we show how the physical dimensions of the variables that govern a problem can be used to find physical laws. Surprisingly, while most engineers learn about dimensional analysis, this topic is not covered explicitly in many science curricula.

2.1 Two rules for physical dimensions

In physics every physical parameter is associated with a physical dimension. The value of each parameter is measured with a certain physical unit. For example, when I measure how long a table is, the result of this measurement has dimension "length". This length is measured in a certain unit, that may be meters, inches, furlongs, or whatever length unit I prefer to use. The result of this measurement can be written as

$$l = 3 \text{ m.}$$  \hfill (2.1)
The variable \( I \) has the physical dimension of length, in this chapter we write this as

\[
I \sim [L].
\]  

(2.2)

The square brackets are used in this chapter to indicate a physical dimension. The capital letter \( L \) denotes length, \( T \) denotes time, and \( M \) denotes mass. Other physical dimensions include electric charge and temperature. When dealing with physical dimensions two rules are useful. The first rule is:

**Rule 1** When two variables are added, subtracted, or set equal to each other, they must have the same physical dimension.

In order to see the logic of this rule we consider the following example. Suppose we have an object with a length of 1 meter and a time interval of one second. This means that

\[
l = 1 \text{ m},
\]

\[
t = 1 \text{ s}.
\]

(2.3)

Since both variables have the same numerical value, we might be tempted to declare that

\[
l = t.
\]

(2.4)

It is, however, important to realize that the physical units that we use are arbitrary. Suppose, for example, that we had measured the length in feet rather than meters. In that case the measurements (2.3) would be given by

\[
l = 3 \text{ ft},
\]

\[
t = 1 \text{ s}.
\]

(2.5)

Now the numerical value of the same length measurement is different! Since the choice of the physical units is arbitrary, we can scale the relation between variables of different physical dimensions in an arbitrary way. For this reason these variables cannot be equal to each other. This implies that they cannot be added or subtracted either.

The first rule implies the following rule.

**Rule 2** Mathematical functions can act on dimensionless numbers only.

To see this, let us consider an example the function \( f(\xi) = e^\xi \). Using a Taylor expansion, this function can be written as:

\[
f(\xi) = 1 + \xi + \frac{1}{2!}\xi^2 + \cdots
\]

(2.6)

According to rule 1 the different terms in this expression must have the same physical dimension. The first term (the number 1) is dimensionless, hence all the other terms in the series must be dimensionless. This means that \( \xi \) must be a dimensionless number as well. This argument can be used for any function \( f(\xi) \) whose Taylor expansion contains different powers of \( \xi \). Note that the argument would not hold for a function such as \( f(\xi) = \xi^2 \) that contains only one power of \( \xi \). To please the purists, rule 2 could easily be reformulated to exclude these special cases.

These rules have several applications in mathematical physics. Suppose we want to find the physical dimension of a force, as expressed in the basic dimensions mass, length, and time. The only thing we need to do is take one equation that contains a force. In this case Newton's law \( F = ma \) comes to mind. The mass \( m \) has physical dimension \([M]\), while the acceleration has dimension \([L/T^2]\). Rule 1 implies that force has the physical dimension \([ML/T^2]\).

**Problem a** The force \( F \) in a linear spring is related to the extension \( x \) of the spring by the relation \( F = -kx \). Show that the spring constant \( k \) has dimension \([M/\ell^2]\).

**Problem b** The angular momentum \( L \) of a particle with momentum \( p \) at position \( r \) is given by

\[
L = r \times p.
\]

(2.7)

where \( \times \) denotes the cross-product of two vectors. Show that angular momentum has the dimension \([ML^2/T]\).

**Problem c** A plane wave is given by the expression

\[
u(\mathbf{r}, t) = e^{i(kr - \omega t)}.
\]

(2.8)

where \( \mathbf{r} \) is the position vector and \( t \) denotes time. Show that \( k \sim [L^{-1}] \) and \( \omega \sim [T^{-1}] \).

In quantum mechanics the behavior of a particle is characterized by a wave equation, that is called the Schrödinger equation. In one space dimension this equation is given by

\[
\frac{\partial^2 \psi}{\partial t^2} = -\frac{\hbar^2}{2m}\frac{\partial^2 \phi}{\partial x^2} + V(x)\psi,
\]

(2.9)

where \( x \) denotes the position, \( t \) denotes the time, \( m \) the mass of the particle, and \( V(x) \) the potential energy of the particle. At this point it is not clear what the wave
2.2 A trick for finding mistakes

The requirement that all terms in an equation have the same physical dimension is an important tool for spotting mistakes. Cipra [26] gives many useful tips for spotting errors in his delightful book "Mistakes [sic] . . . and how to find them before the teacher does." As an example of using dimensional analysis for spotting mistakes, we consider the erroneous equation

$$ E = mc^2, \quad (2.10) $$

where $E$ denotes energy, $m$ denotes mass, and $c$ is the speed of light. Let us first find the physical dimension of energy. The work done by a force $F$ over a displacement $d\mathbf{r}$ is given by $dE = F \cdot d\mathbf{r}$. We showed in Section 2.1 that force has the dimension $[ML^2/T^2]$. This means that energy has the dimension $[ML^2/T^2]$. The speed of light in the right-hand side of expression (2.10) has dimension $[L/T]$, which means that the right-hand side has physical dimension $[ML^2/T^2]$. This is not an energy, which has dimension $[ML^2/T^2]$. Therefore expression (2.10) is wrong.

Problem a

Now that we have determined that expression (2.10) is incorrect we can use the requirement that the dimensions of the different terms must match to guess how to set it right. Show that the right-hand side must be divided by a velocity to match the dimensions.

It is not clear that the right-hand side must be divided by the speed of light to give the correct expression $E = mc^2$. Dimensional analysis tells us only that it must be divided by something with the dimension of velocity. For all we know, it could be the speed at which the average snail moves.

Problem b

Is the following equation dimensionally correct?

$$ (v \cdot \nabla) = -\nabla p. \quad (2.11) $$

In this expression v is the velocity of fluid flow, p is the pressure, and $\nabla$ is the gradient vector (which essentially is a derivative with respect to the space coordinates). You can use that pressure has the dimension is force per unit area.

Problem c

Answer the same question for the expression that relates the particle velocity $v$ to the pressure $p$ in an acoustic medium:

$$ v = \frac{p}{\rho c} \quad (2.12) $$

Here $\rho$ is the mass density and $c$ is velocity of propagation of acoustic waves.

Problem d

In quantum mechanics, the energy $E$ of the harmonic oscillator is given by

$$ E_n = \hbar \omega (n + 1/2), \quad (2.13) $$

where $\omega$ is a frequency, $n$ is a dimensionless integer, and $\hbar$ is Planck’s constant divided by $2\pi$ as introduced in problem d of the previous section. Verify if this expression is dimensionally correct.

In general it is a good idea to carry out a dimensional analysis while working in mathematical physics because this may help in finding the mistakes that we all make while doing derivations. It takes a little while to become familiar with the dimensions of properties that are used most often, but this is an investment that pays off in the long run.

2.3 Buckingham pi theorem

In this section we introduce the Buckingham pi theorem. This theorem can be used to find the relation between physical parameters based on dimensional arguments. As an example, let us consider a ball shown in Figure 2.1 with mass $m$ that is dropped from a height $h$. We want to find the velocity with which it strikes the ground. The potential energy of the ball before it is dropped is $mgh$, where $g$ is the acceleration of gravity. This energy is converted into kinetic energy $\frac{1}{2}mv^2$ as it strikes the ground. Equating these quantities and solving for the velocity gives:

$$ v = \sqrt{2gh}. \quad (2.14) $$

Now let us suppose we did not know about classical mechanics. In that case, dimensional analysis could be used to guess relation (2.14). We know that the velocity is some function of the acceleration of gravity, the initial height, and the mass of the particle: $v = f(g, h, m)$. The physical dimensions of these properties
The treatment given here may appear to be cumbersome. This analysis, however, can be carried out in a systematic fashion using the Buckingham pi theorem [23] which states the following:

**Buckingham pi theorem** If a problem contains \( N \) variables that depend on \( P \) physical dimensions, then there are \( N - P \) dimensionless numbers that describe the physics of the problem.

The original paper of Buckingham is very clear, but as we will see at the end of this section, this theorem is not fool-proof. Let us first apply the theorem to the problem of the falling ball. We have four variables: \( v, g, h, \) and \( m \), so that \( N = 4 \). These variables depend on the physical dimensions \( [M], [L], \) and \( [T] \), hence \( P = 3 \). According to the Buckingham pi theorem, \( N - P = 1 \) dimensionless number characterizes the problem. We want to express the velocity in the other parameters; hence we seek a dimensionless number of the form

\[
v g^{\alpha} h^{\beta} m^{\gamma} \sim 1\]  

where the notation in the right-hand side means that it is dimensionless. Let us seek the exponents \( \alpha, \beta, \) and \( \gamma \) that make the left-hand side dimensionless. Inserting the dimensions of the different variables then gives the following dimensions

\[
\left[ \frac{L}{T} \right], \left[ \frac{L^2}{T^2} \right], \left[ M^0 \right] \sim 1.
\]

The left-hand side depends on length as \( [L^{1+\alpha+\beta}] \). The left-hand side can only be independent of length when the exponent is equal to zero. Applying the same reasoning to each of the dimensions length, time, and mass, then gives

\[
\begin{align*}
\text{dimension } [L]: & \quad 1 + \alpha + \beta = 0, \\
\text{dimension } [T]: & \quad -1 - 2\alpha = 0, \\
\text{dimension } [M]: & \quad \gamma = 0.
\end{align*}
\]

This constitutes a system of three equations with three unknowns.

**Problem a** Show that the solution of this system is given by

\[
\alpha = \beta = -\frac{1}{2}, \quad \gamma = 0.
\]

Inserting these values into expression (2.18) shows that the combination \( v g^{-1/2} h^{-1/2} \) is dimensionless. This implies that

\[
v = C \sqrt{gh},
\]
where $C$ is the one dimensionless number in the problem as dictated by the Buckingham $pi$ theorem.

The approach taken here is systematic. In his original paper [23], Buckingham applied this treatment to a number of problems: the thrust provided by the screw of a ship, the energy density of the electromagnetic field, the relation between the mass and radius of the electron, the radiation of an accelerated electron, and heat conduction.

There is, however, a catch that we introduce with an example. When air (or water) has a stably stratified mass–density structure, it can support oscillations where the restoring force is determined by the density gradient in the air. These oscillations occur with the Brunt–Väisälä frequency $\omega_B$ given by [80, 82]:

$$\omega_B = \sqrt{\frac{g}{N}} \frac{\partial \theta}{\partial z}.$$  \hspace{1cm} (2.23)

In this expression, $g$ is the acceleration of gravity, $z$ is height, and $\theta$ is potential temperature (a measure of the thermal structure of the atmosphere).

**Problem b** Verify that this expression is dimensionally correct.

**Problem c** Check that this expression is also dimensionally correct when $\theta$ is replaced by the air pressure $p$, or the mass density $\rho$.

The result of problem c indicates that the potential temperature $\theta$ can be replaced by any physical parameter, and expression (2.23) is still dimensionally correct. This means that a dimensional analysis alone can never be used to prove that $\theta$ should be the potential temperature. In order to show this we need to know more of the physics of the problem.

Another limitation of the Buckingham $pi$ theorem as formulated in its original form is that the theorem assumes that physical parameters need to be multiplied or divided to form dimensionless numbers; see equation (3) of reference [23]. The derivative of one variable with respect to another, however, has the same dimension as the ratio of these variables. Consider for example a problem where dimensional analysis shows that the variable of interest depends on the ratio of the acceleration of gravity and the height: $g/h$. The derivative of $g$ with height $dg/dz$ has the same physical dimension as $g/h$. Therefore, a dimensional analysis alone cannot completely describe the physics of the problem. Nevertheless, as we will see in the following section, it may provide valuable insights.

In this section we study the lift of a wing. Since in stationary flight the lift provided by a wing is equal to the weight of the aircraft or bird that is carried by the wing, we denote the lift of the wing with the symbol $W$. Since the lift is a force, this quantity has the dimension force: $W \sim [F] = [ML/T^2]$. The lift depends on the mass density $\rho$ of the air, the velocity $v$ of the air, and the surface area $S$ of the wing.

**Problem a** Show that $\rho \sim [M/L^3]$, $v \sim [L/T]$, and $S \sim [L^2]$.

**Problem b** Count the number of variables and number of physical dimensions to show that in the jargon of the Buckingham $pi$ theorem $N = 4$ and $P = 3$.

This means that there is $N - P = 1$ dimensionless number that characterizes the lift of the wing. We want to express the lift $W$ in the other parameters, therefore we seek a dimensionless number of the form

$$W \rho^a v^b S^c \sim [1].$$  \hspace{1cm} (2.24)

**Problem c** Show that the requirement that the left-hand side does not depend on mass, length, and time, respectively, leads to the following linear equations:

$$1 + a = 0,$$

$$1 - 3a + \beta + 2y = 0,$$

$$2 + \beta = 0.$$  \hspace{1cm} (2.25)

**Problem d** Solve this system to derive that

$$a = y = -1, \quad \beta = -2.$$  \hspace{1cm} (2.26)

Inserting this result in expression (2.24) implies that $W/(\rho v^2 S)$ is a dimensionless number. When this constant is denoted by $C_L$, this means that the lift is given by

$$W = C_L \rho v^2 S.$$  \hspace{1cm} (2.27)

The coefficient $C_L$ is called the lift coefficient [55]. This coefficient depends on the shape of the wing, and on the angle of attack. (This is a measure of the orientation of the wing to the airflow.) Let us think about the solution (2.27) for a moment. This expression states that the lift is proportional to the surface area; this makes sense: a larger wing produces more lift. The lift depends on the square of the velocity, it
stands to reason that a larger flow velocity gives a larger lift, but that the lift increases quadratically with the velocity is not easy to see. Lastly, the lift is proportional to the mass density of the air: for a given velocity heavier air provides a larger lift because the airflow deflected by the wing has a larger momentum.

This has implications for the design of airports. For example, the airport of Denver is located at an elevation of about 1,600 meters. This high elevation, in combination with the warm temperatures in summertime, leads to a relatively small mass density of the air. Since the surface area of the wings of aircraft is fixed by their design, the relatively small mass density can be compensated by a larger take-off velocity $v$ only. In order to achieve this large take-off velocity, aircraft need a longer runway to accelerate to the required take-off velocity. For this reason, the airport in Denver has extra long runways. All these conclusions follow from dimensional analysis only.

### 2.5 Scaling relations

We can take the dimensional analysis of the previous section even a step further. Suppose we consider different flying objects, and that each object is characterized by a linear dimension $l$.

**Problem a** Use dimensional arguments to show that the volume $V$ scales with the size as $V \sim l^3$, and that the surface area scales as $S \sim l^2$. (The volume $V$ should not be confused with the velocity $v$.)

**Problem b** Show that this implies that

$$S \sim V^{2/3}. \tag{2.28}$$

The mass of the flying object is proportional to its mass density $\rho_x$ by the relation $m = \rho_x V$. The lift required to support this mass is given by

$$W = g\rho_x V. \tag{2.29}$$

**Problem c** Insert the relations (2.28) and (2.29) into expression (2.27) to show that

$$g\rho_x V^{1/3} = C_4 \rho v^2. \tag{2.30}$$

### 2.6 Dependence of pipe flow on the radius of the pipe

**Problem d** Solve this expression for $V$, and insert this result into expression (2.29) to derive the following relation between the lift and the velocity:

$$W = \frac{C_4 \rho v^3}{\frac{2}{2}}. \tag{2.31}$$

This expression predicts that the lift varies with the velocity to the sixth power. Figure 2.2 shows a compilation of the weight versus the cruising speed for various aircraft (top right), birds (middle), insects, and butterflies (bottom left). This figure is reproduced from the wonderful book of Tennekes [196] about the science of flight. The points in this figure cluster around a straight line. The weight and the cruising speed are shown on a double logarithmic scale; hence the straight line implies a power law relation of the form $W \sim v^n$.

**Problem e** Measure the slope of the line in Figure 2.2 and show that this slope is close to the value $n = 6$ predicted by expression (2.31).

Note that the lift in Figure 2.2 ranges over 11 orders of magnitude. Despite this extreme range in parameter values, the scaling law (2.31) holds remarkably well. The individual points show departures from the scaling law. The reason is that the density $\rho_x$ and the lift coefficient $C_4$ vary among different flying objects; the shape of the wing of a Boeing 747 is different from the shape of the wing of a butterfly.

This example shows that dimensional arguments can be useful in explaining the relationship between different physical parameters. Such relationships are also of importance in the design of scale experiments. An example of a scale experiment is a model of an aircraft in a wind tunnel. All physical parameters need to be scaled appropriately with the size of the model aircraft so that the physics is unaltered by the scaling. This is the case when the dimensionless numbers determined with the Buckingham pi theorem are the same for the scaled model as for the real aircraft. In this way the Buckingham pi theorem provides a systematic procedure for the design of scale experiments as well [23].

### 2.6 Dependence of pipe flow on the radius of the pipe

The flow of a viscous fluid through a porous medium is important for understanding and managing aquifers and hydrocarbon reservoirs. Here we use dimensional analysis to study the dependence of flow of a viscous fluid through a cylindrical pipe as shown in Figure 2.3. The flow is driven by a pressure gradient $\delta p / \delta x$ along
2.6 Dependence of pipe flow on the radius of the pipe

Fig. 2.3 The geometry of a pipe through which fluid flows.

Problem a The physical quantities that are of relevance to this problem are the pressure gradient $\frac{dp}{dx}$, the viscosity $\mu$, the radius $R$, and the flow rate $\Phi$. Write down the physical dimensions of each of these properties. In order to find the dimension of the viscosity you can use the relation $\tau = \mu \frac{dv}{dz}$, where $\tau$ is the shear stress (with dimension pressure), $v$ the velocity, and $z$ distance.

Problem b Use the Buckingham pi theorem to show that the flow rate is given by

$$\Phi = \text{constant} \frac{\frac{dp}{dx}}{\mu} R^4.$$  (2.32)

Problem c This expression states that the flow rate is proportional to the pressure gradient, which reflects the fact that a stronger pressure gradient generates a stronger driving force for the flow, and hence a stronger flow. Give a similar physical explanation for the dependence of the flow rate on the viscosity and the radius. At first you might think that the flow rate is proportional to the surface area $\pi R^2$ of the pipe. Try to give a physical explanation for the $R^4$-dependence of the flow rate on the radius.

The result (2.32) can also be obtained by solving the Navier–Stokes equation (11.55) for the appropriate boundary condition, and by integrating the flow velocity over the pipe to give the flow rate $\Phi$. This treatment is more cumbersome than the analysis of this section, but it does provide the proportionality constant in expression (2.32).

Fig. 2.2 The weight of many flying objects (vertical axis) against their cruising speed (horizontal axis) on a log-log plot. This figure is reproduced from reference [100] with permission from MIT Press.

the center axis of the cylinder. We assume that the fluid has a viscosity $\mu$, and we want to find the relation between the strength of the flow along the pipe per unit time and the radius $R$. As a measure of the flow rate we use the velocity of the flow per unit time, and designate this quantity with the symbol $\Phi$. 