Fun with complex numbers

1. Express the following in “Cartesian form” \(x + iy\), where \(x\) and \(y\) are real:
   \[
   \frac{1}{(2 - 3i)}, \ (1 + 2i)/(3 + 4i), \ 5e^{6i}.
   \]

2. Express the following in “polar form” \(re^{i\varphi}\), where \(r\) is a real positive number and \(\theta\) is real:
   \[-6, \ -5i, \ (1 + i)/\sqrt{2}, \ 2 - 3i, \ (2 + i)/(1 + 2i).\]

3. (i) Find all the cube roots of \(-1\), i.e. \((-1)^{1/3}\), and express them all in both polar form and in Cartesian form. (ii) Plot and label them in the complex plane.

4. Show that there are infinitely many values of \(i^i\) and they are all real. (Hint: Remember the definition of the complex exponential: \(w^z = \exp(z \ln w)\).)

5. Prove the trigonometric identities for \(\cos(a + b)\) and \(\sin(a + b)\) by taking the real and imaginary parts of the identity \(\exp(i(a + b)) = \exp(ia) \exp(ib)\). You may of course use the fact that \(\exp(i\theta) = \cos \theta + i \sin \theta\).

6. Show that the complex conjugate operation that sends \(z = x + iy\) to \(z^* = x - iy\) enjoys the following properties:
   \[
   (z + w)^* = z^* + w^* \\
   (zw)^* = z^*w^* \\
   (z/w)^* = z^*/w^* \\
   (e^z)^* = e^{z^*} \\
   zz^* = |z|^2
   \]
   where \(|z| = \sqrt{x^2 + y^2}\) is the modulus of \(z\).