Homework 4

Solving exercises is the most effective way of learning physics. Although only one third of the final grades for this course will be based on the homeworks, you should take them very seriously.

Assignment to be turned in at the beginning of the class on Thursday, April 20 by students registered to the course:

• Work the three exercises below

Exercises: Radiation from accelerated masses

We consider the gravitational radiation produced during the elastic deflection of a particle by a fixed scattering center. We denote the initial and final four-momenta of the particle with $p^\mu$ and $p'^\mu$. The energy-momentum tensor of the particle is

$$T^{\mu\nu}(x) = \frac{p^\mu p'^\nu}{\gamma m} \delta^{(3)}(x - x_0(t)),$$  \hspace{1cm} (1)

where $m$ is the particle’s mass, $\gamma = 1/(1 - v^2/c^2)$ and $x_0(t)$ is the particle’s trajectory. Approximating the collision as instantaneous, the Fourier transform of the energy-momentum tensor for a massive particle, Eq. (1), reads

$$\tilde{T}^{\mu\nu}(\omega, k) = \frac{c}{2 \gamma m} \left[ \frac{p^\mu p'^\nu}{\omega - k \cdot v} - \frac{p'^\mu p'^\nu}{\omega - k \cdot v'} \right].$$ \hspace{1cm} (2)

1. Elastic collisions: non-relativistic limit [M. Maggiore (2006)] (3.5 points)

Show that in the non-relativistic limit, Eq. (1) reduces to

$$\tilde{T}^{ij}(\omega) \simeq \frac{ic}{\omega m} \left( p_i p_j - p'_i p'_j \right).$$ \hspace{1cm} (3)

Assume $v = v(1, 0, 0)$ and $v' = v(\cos \theta_s, \sin \theta_s, 0)$ and $|v| = |v'|$ (elastic collision). In class we derived that the distribution of the energy radiated in gravitational waves is ($d\Omega = d\cos \theta d\phi$)

$$\frac{dE}{d\Omega d\omega} = \frac{G m^2 v^4}{2 \pi^2 c^5} \Lambda_{ij,kl}(n) \int_0^{+\infty} d\omega \omega^2 \tilde{T}_{ij}(\omega, \omega n/c) \tilde{T}_{kl}(\omega, \omega n/c),$$ \hspace{1cm} (4)

where

$$n = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad \Lambda_{ij,kl} = P_{ik} P_{jl} - P_{ij} P_{kl}/2 \quad P_{ij} = \delta_{ij} - n_i n_j.$$ \hspace{1cm} (5)

Plugging Eq. (3) into Eq. (4), derive the following formula

$$\frac{dE}{d\Omega d\omega} = \frac{G m^2 v^4}{\pi^2 c^5} \left[ f_1(\theta_s) - f_2(\theta_s; \phi) \sin^2 \theta + f_3(\theta_s; \phi) \sin^4 \theta \right].$$ \hspace{1cm} (6)

What are the explicit expressions for the coefficients $f_1$, $f_2$ and $f_3$? Show that the radiation is mostly emitted along the $z$-axis. Evaluate $dE/d\omega$.

2. Elastic collisions: relativistic limit [M. Maggiore (2006)] (3 points)

In the relativistic limit $p = \gamma m v$. Setting $k = n \omega/c$, we have

$$\tilde{T}^{ij}(\omega, n \omega/c) = \frac{c \gamma m}{i \omega} \left( \frac{v_i v_j}{1 - n \cdot v/c} - \frac{v'_i v'_j}{1 - n \cdot v'/c} \right).$$ \hspace{1cm} (7)
Plugging Eq. (7) into Eq. (4), derive the following formula

$$\frac{dE}{d\Omega d\omega} = \frac{G m^2 \gamma^2 v^4}{\pi^2} \left[ f_1(v, \theta_s; \theta, \phi) - f_2(v, \theta_s; \theta, \phi) \sin^2 \theta + f_3(v, \theta_s; \theta, \phi) \sin^4 \theta \right].$$  \hspace{1cm} (8)

What are the explicit expressions for the coefficients $f_1$, $f_2$ and $f_3$? Notice that the factors $1/(1 - n \cdot v/c)$ appearing in the coefficients $f_1$, $f_2$ and $f_3$ tend to bend the radiation in the direction of the motion.

3. **Lack of beaming from accelerated masses [M. Maggiore (2006)] (4 points)**

Here we want to compare the GW radiation to the EM radiation and show that in the former case there is a lack of beaming.

We consider a relativistic particle which is instantaneously accelerated from 0 to $v$. The Fourier transform of the energy-momentum tensor reads

$$\tilde{T}^{ij}(\omega, n \omega/c) = \frac{c \gamma m}{i \omega} \left( \frac{v_i v_j}{1 - \cos \theta v/c} \right).$$ \hspace{1cm} (9)

Show that the distribution of the energy radiated in gravitational waves is

$$\left( \frac{dE}{d\Omega d\omega} \right)_{GW} = \frac{G m^2 \gamma^2}{4\pi^2 c} \left( \frac{v}{c} \right)^4 p_{GW}(\theta),$$ \hspace{1cm} (10)

and evaluate the coefficient $p_{GW}(\theta)$. The distribution of the energy radiated in electromagnetic waves is

$$\left( \frac{dE}{d\Omega d\omega} \right)_{EM} = \frac{e^2}{4\pi^2 c} \left( \frac{v}{c} \right)^2 p_{EM}(\theta), \quad p_{EM}(\theta) = \frac{\sin^2 \theta}{(1 - \cos \theta v/c)^2}.$$ \hspace{1cm} (11)

Estimate for which value of $\theta$, the functions $p_{EM}(\theta)$ and $p_{GW}(\theta)$ have a maximum. Estimate also the width of the maximum. From these results you should conclude that even in the limit $\gamma \to \infty$ the gravitational radiation is not beamed in a narrow cone, as the electromagnetic radiation does, but it is extended over a solid angle comparable to $4\pi$. 
