Homework 1 Solutions

Problem 1: Electromagnetic Field

The idea behind these problems is to “re-derive” some of the known results in electromagnetism using the classical field theory approach, i.e., with the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu}$$

where

$$F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and identifying the electric and magnetic fields as

$$E^i = -F^{0i},$$
$$\varepsilon^{ijk} B^k = -F^{ij}$$

For example, we already showed in lecture that Maxwell’s equations are simply the Euler-Lagrange equations.

a) Energy-momentum

Based on Noether’s theorem, construct the energy-momentum tensor for classical electromagnetism from the above Lagrangian.

Note that the usual procedure does not result in a symmetric tensor. To remedy that, we can add to $T^{\mu \nu}$ a term of the form $\partial_\lambda K^{\lambda \mu \nu}$, where $K^{\lambda \mu \nu}$ is antisymmetric in its first two indices. Such an object is automatically divergenceless, so

$$\hat{T}^{\mu \nu} = T^{\mu \nu} + \partial_\lambda K^{\lambda \mu \nu}$$

is an equally good energy-momentum tensor with the same globally conserved energy and momentum. Show that this construction, with

$$K^{\lambda \mu \nu} = F^{\mu \lambda} A_\nu$$

leads to an energy-momentum tensor $\hat{T}$ that is symmetric and yields the standard (i.e., known without using field theory) formulae for the electromagnetic energy and momentum densities:

$$\mathcal{E} = \frac{1}{2} (E^2 + B^2),$$
$$\mathbf{S} = \mathbf{E} \times \mathbf{B}$$
Solution:

First, we calculate the energy-momentum tensor using

$$T_{\mu}^{\nu} = \frac{\delta L}{\delta(\partial_{\mu}A_{\lambda})} \partial_{\nu}A_{\lambda} - \delta_{\mu}^{\nu}L$$  \hspace{1cm} (9)

Expand the Lagrangian as

$$L = -\frac{1}{4}F_{\mu\nu}F_{\mu\nu} = -\frac{1}{2}(\partial^{\mu}A^{\nu}\partial_{\mu}A_{\nu} - \partial^{\mu}A_{\nu}\partial_{\nu}A_{\mu})$$  \hspace{1cm} (10)

we can calculate

$$\frac{\delta L}{\delta(\partial_{\mu}A_{\lambda})} = -F_{\mu\lambda}$$  \hspace{1cm} (11)

Thus we get

$$T_{\mu\nu} = -F_{\mu\lambda}\partial_{\nu}A_{\lambda} + \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}F_{\rho\sigma}$$  \hspace{1cm} (12)

where we have raised the $\nu$ index using metric $\eta^{\mu\nu}$. This is obviously not symmetric under exchange of $\mu\nu$ indices. To make it a symmetric tensor, we add total derivative term:

$$\partial_{\lambda}K^{\lambda\mu\nu} = (\partial_{\lambda}F_{\mu\lambda})A^{\nu} + F_{\mu\lambda}(\partial_{\lambda}A^{\nu})$$  \hspace{1cm} (13)

We know from equation of motion that $\partial_{\lambda}F^{\mu\lambda} = 0$. Therefore

$$\hat{T}_{\mu\nu} = F_{\mu\lambda}F_{\nu}^{\lambda} + \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}F_{\rho\sigma}$$  \hspace{1cm} (14)

which is manifestly symmetric in $\mu\nu$ indices. Now we can express it in terms of physical electric and magnetic fields. The energy density is given by

$$\epsilon = \hat{T}^{00} = F^{0i}F_{i}^{\;0} + \frac{1}{4}(2F^{0i}F_{0i} + F^{ij}F_{ij})$$  \hspace{1cm} (15)

$$= \frac{1}{2}F^{0i}F_{0i} + \frac{1}{4}F^{ij}F_{ij}$$

$$= \frac{1}{2}(|\vec{E}|^2 + |\vec{B}|^2)$$

where in the last equality we used $\epsilon^{ijk}\epsilon^{ijl} = 2\delta^{kl}$. Similarly, the momentum density is given by

$$\vec{S} = \hat{T}^{0i} = F^{0k}F_{k}^{\;i} = -E^{k}\epsilon^{kil}B^{l} = (\vec{E} \times \vec{B})^{i}$$  \hspace{1cm} (16)

b) Subtlety with going to Hamiltonian formalism

Exercises 2.4 and 2.5 of Lahiri and Pal.

Due to this subtlety, we will not quantize electromagnetic field to begin with (even though historically it was the first QFT). We will return to this issue when we quantize the electromagnetic field later in the course.
Solution to Exercise 2.4

First, we need to find terms in the Lagrangian with time derivative of fields $A^\mu$:

\[
- \frac{1}{4} F^{\mu\nu} F_{\mu\nu} = -\frac{1}{4} \left( 2 F^{0i} F_{0i} + F^{ij} F_{ij} \right) \\
= \frac{1}{2} [ (\dot{A}^i)^2 + (\nabla A^0)^2 - 2 \dot{A}^i \partial^i A_0 ] - \frac{1}{4} F^{ij} F_{ij}
\]

(17)

The canonical momenta are

\[
\Pi_0 \equiv \frac{\delta L}{\delta \dot{A}^0} = 0 \\
\Pi_i \equiv \frac{\delta L}{\delta \dot{A}^i} = \dot{A}^i - \partial^i A_0
\]

(18)

(19)

We can see that from the above equations we cannot solve for $\dot{A}^0$. The reason is that there is no term in the Lagrangian with time derivative of $A^0$. In other words, $A^0$ is not a dynamical field.

Solution to Exercise 2.5

Now, if we fix the gauge by choosing $A^0 = 0$, and treat $A^i$ as dynamical fields, we get

\[
\Pi_i \equiv \frac{\delta L}{\delta \dot{A}^i} = \dot{A}^i
\]

(20)

Obviously, it can be inverted to solve for $\dot{A}^i$.

Problem 2: Real, free scalar/Klein-Gordon Field

This is the simplest classical field theory and so the first one that we will quantize. For the Lagrangian

\[
\mathcal{L} = \frac{1}{2} \left( \partial^\mu \phi \right) \left( \partial_\mu \phi \right) - \frac{1}{2} m^2 \phi^2
\]

(21)

where $\phi$ is a real-valued field,

(i) Show that the Euler-Lagrange equation is the Klein-Gordon equation for the field $\phi$.

(ii) Find the momentum conjugate to $\phi(x)$, denoted by $\Pi(x)$.

(iii) Use $\Pi(x)$ to calculate the Hamiltonian density, $\mathcal{H}$.

(iv) Based on Noether’s theorem, calculate the stress-energy tensor, $T^\mu_\nu$, of this field and the conserved charges associated with time and spatial translations, i.e., the energy-momentum, $P^\mu$, of this field.

(v) Using the Euler-Lagrange (i.e., Klein-Gordon) equation, show that $\partial_\mu T^\mu_\nu = 0$ for this field. (Of course, this result was expected from Noether’s theorem.)
(vi) Finally, show that $P^0$ that you calculated above in part (iv) is the same as the total Hamiltonian, i.e., spatial integral of $\mathcal{H}$ which you calculated above in part (iii).

We will determine eigenstates/values of this (total) Hamiltonian when we quantize the field. And, $P^i$ can be interpreted as the (physical) momentum carried by the field (not to be confused with canonical momentum!). This $P_i$ will be used in interpreting the eigenstates of the Hamiltonian of the quantized scalar field.

**Solutions:**

(i) Euler-Lagrange equation for $\phi$

\[
\frac{\delta \mathcal{L}}{\delta \phi} = \partial_\mu \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)}
\]  
(22)

Substituting in the Lagrangian,

\[
-m^2 \phi = \partial_\mu (\partial_\mu \phi)
\]  
(23)

\[
\Rightarrow (\partial^2 + m^2) \phi = 0
\]  
(24)

which is the Klein-Gordon equation.

(ii)

\[
\Pi(x) = \frac{\delta \mathcal{L}}{\delta \dot{\phi}} = \dot{\phi}
\]  
(25)

(iii)

\[
\mathcal{H} = \Pi \dot{\phi} - \mathcal{L} = \frac{1}{2} [\dot{\phi}^2 + (\vec{\nabla} \phi)^2 + m^2 \phi^2]
\]  
(26)

(iv)

\[
T^{\mu \nu} = \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)} \partial_\nu \phi - \eta^{\mu \nu} \mathcal{L}
\]  
(27)

\[
= \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} \eta^{\mu \nu} [\partial^\rho \phi \partial_\rho \phi - m^2 \phi^2]
\]

The conserved charge is given by,

\[
P^\mu = \int d^3 x \ T^{0 \mu}
\]  
(28)

(v) The divergence of the stress-energy tensor,

\[
\partial_\mu T^\mu_\nu = \partial_\mu (\partial^\mu \phi \partial^\nu \phi - \frac{1}{2} \eta^{\mu \nu} [\partial^\rho \phi \partial_\rho \phi - m^2 \phi^2])
\]  
(29)

\[
= \partial^2 \phi \partial^\nu \phi + \partial^\mu \phi \partial_\mu \partial^\nu \phi - \frac{1}{2} \partial^\nu [\partial^\rho \phi \partial_\rho \phi - m^2 \phi^2])
\]  
(30)

\[
= \partial^2 \phi \partial^\nu \phi + \partial^\mu \phi \partial_\mu \partial^\nu \phi - [\partial^\rho \phi \partial_\rho \phi + \partial^\rho \phi \partial_\rho \phi - m^2 \phi \partial^\nu \phi]
\]  
(31)

\[
= (\partial^2 + m^2) \phi \partial^\nu \phi = 0
\]  
(32)
Therefore, if the field satisfies its equation of motion (the Klein-Gordon equation in this case), the stress-energy tensor is conserved. Therefore, Noether current conservation relies on the equations of motion which are satisfied for a classical field.

(vi) Using the expression above for $P^\mu$, we get

$$P^0 = \int d^3x \left[ \frac{1}{2} \dot{\phi}^2 + (\nabla \phi)^2 + m^2 \phi^2 \right] = \int d^3x \ H$$

$$P^i = \int d^3x \ \dot{\phi} \partial^i \phi$$

Problem 3: Scale invariance

Exercise 2.10 of Lahiri and Pal.

The transformations involve a simultaneous re-scaling of the coordinates and the fields, hence the name “scale invariance” given to this symmetry.

Solution:

The transformations in Lahiri and Pal and those in Peskin and Schroeder follow different conventions. They are potentially quite confusing, so it is a good idea to keep one convention handy. We will use the Lahiri and Pal notation here.

The transformation is

$$x \rightarrow x' = bx$$

$$\phi(x) \rightarrow \phi'(x') = \frac{\phi(x)}{b}$$

It is important to note that in this convention, the argument of the field (in the right-most expression) does not change with the transformation. As a reference for this convention, one can remember that the scalar transforms like $\phi(x) \rightarrow \phi(x)$ under a Lorentz transformation.

The infinitesimal version of the transformation is given by

$$x \rightarrow (1 + \epsilon)x \quad \delta x_\mu = \epsilon x_\mu$$

$$\phi(x) \rightarrow 1/(1 + \epsilon)\phi(x) \quad \delta \phi = -\epsilon \phi$$

Again, remember that for Lorentz transformations on a scalar field, $\delta \phi$ would be zero in this convention.

The Lagrangian is given by,

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \lambda \phi^4$$

Under the transformation, $\partial \rightarrow \frac{1}{b} \partial$. Therefore, the transformed Lagrangian becomes,

$$\mathcal{L}_b = \frac{1}{b^4} \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{b^4} \phi^4$$

$$= \frac{1}{b^4} \mathcal{L}$$
We can now look at the transformation of the action under this symmetry,

\[ \int d^4x \, L \rightarrow \int b^4 \, d^4x \, L_b = \int d^4x \, L \]  

Therefore, the action is invariant under this symmetry. The Noether current is,

\[ \epsilon J^\mu = \frac{\partial L}{\partial (\partial_\mu \phi)} \delta \phi - T^{\mu\nu} \delta x_\nu \]  

\[ = \frac{\partial L}{\partial (\partial_\mu \phi)} \delta \phi - \left( \frac{\partial L}{\partial (\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} L \right) \delta x_\nu \]  

\[ = -\epsilon \partial_\mu \phi - \left( \partial_\mu \phi \partial^\nu \phi - g^{\mu\nu} \left( \frac{1}{2} \partial_\rho \phi \partial^\rho \phi - \lambda \phi^2 \right) \right) \epsilon x_\nu \]  

**Problem 4: Complex scalar/Klein Gordon field coupled to electromagnetism (Scalar electrodynamics)**

Exercises 2.9 (b) and (c) of Lahiri and Pal. Neglect the potential term, \( V (\phi^\dagger \phi) \), given in the Lagrangian in exercise 2.3 of Lahiri and Pal for these problems.

The free complex Klein-Gordon field was discussed in lecture. In particular, it was already shown that the Euler-Lagrange equation is the Klein-Gordon equation (exercise 2.3 of Lahiri and Pal) and the conserved current corresponding to the transformation \( \phi \rightarrow e^{i\alpha} \phi \) was already calculated [exercise 2.9 (a) of Lahiri and Pal] so that there is no need to do it again here. This field is a simple generalization of the case of a real field so that it will be the second field to be quantized.

The purpose of exercises 2.9 (b) and (c) in Lahiri and Pal is to study the addition of an interaction of this field with the electromagnetic field. We will return to quantization of this theory later in the course.

**Solution:**

b) The Lagrangian we are asked to assume is

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \left[ (\partial^\mu - iqA^\mu) \phi \right] \left[ (\partial_\mu + iqA_\mu) \phi \right] - m^2 \phi^\dagger \phi \]  

The infinitesimal transformations are,

\[ \delta \phi = -iq \theta \phi \]  

\[ \delta \phi^\dagger = iq \theta \phi^\dagger \]  

The Noether current,

\[ \theta J^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^\dagger)} \delta \phi^\dagger \]  

\[ = \left[ (\partial^\mu - iqA^\mu) \phi \right] (-iq \theta \phi) + \left[ (\partial_\mu + iqA_\mu) \phi \right] (iq \theta \phi^\dagger) \]  

\[ = -iq \theta \left[ \phi \partial^\mu \phi^\dagger - \phi^\dagger \partial_\mu \phi - 2iqA^\mu \phi \phi^\dagger \right] \]
c) The Euler-Lagrange equation for $A^\mu$

$$\frac{\partial \mathcal{L}}{\partial (A^\mu)} = \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\nu A^\mu)}$$  \hspace{1cm} (51)$$

Note that

$$\frac{\partial (F_{\alpha\beta}F^{\alpha\beta})}{\partial (\partial_\nu A^\mu)} = (g_{\alpha\nu}g_{\beta\mu} - g_{\alpha\mu}g_{\beta\nu})F^{\alpha\beta} + (\delta^\alpha_\nu \delta^\beta_\mu - \delta^\alpha_\mu \delta^\beta_\nu)F_{\alpha\beta}$$

$$= 4F_{\nu\mu}$$  \hspace{1cm} (52)$$

Therefore,

$$\partial_\nu F_\mu^\nu = -i q \phi^\dagger \left[ (\partial_\mu + i q A_\mu) \phi \right] + i q A_\mu \phi \left[ (\partial_\mu - i q A_\mu^\dagger \right]$$  \hspace{1cm} (53)$$

The term on the right hand side is simply the Noether current derived above.

$$\partial_\nu F^{\nu\mu} = j^\mu$$  \hspace{1cm} (54)$$