Illustrating $K^\mu M_\mu = 0$ using "scalar" Compton scattering

Scalar QED: $L = (D^\mu \phi)^+ (D^\mu \phi) - m^2 \phi^+ \phi - \frac{i}{4} F_{\mu \nu} F^{\mu \nu}$

where $D_\mu = \partial_\mu + i e A_\mu$, i.e., $\phi$ destroys spin-0 particle of charge $-1$ (or creates spin-0 particle of charge $+1$)...

Feynman rules: canonical quantization is a bit tricky due to derivative interaction, i.e., at least $\partial_\mu \phi^+ i e A_\mu \phi \quad \Rightarrow \quad \text{(naively), expression for conjugate momentum of } \phi, \text{ i.e., } \partial X / \partial (\partial_\mu \phi) \text{ is modified } \}$: easier to do it using functional integral formalism (see problem 9.1 in Peskin & Schroeder) with the result:

$$\gamma^\mu \gamma^\nu = -i e (p + p')^\mu \quad 2 \gamma^\mu = 2 i e^2 g^\mu_\nu$$

Note (i) (arrow on scalar line in 1st vertex denotes flow of negative charge and (ii) factor of 2 in 2nd vertex is a combinatorial one due to 2 identical particles, i.e., photons, at the same vertex)
"Compton" scattering in scalar QED:

\[ \phi^{+}(p) + \gamma(k, r) \rightarrow \phi^{+}(p') + \gamma(k', r') \]

(spin-0 particle of charge -1)

\(\gamma\) denotes polarization

3 diagrams (2 similar to \(e^- + \gamma \rightarrow e^- + \gamma\), i.e., due to 1st vertex above; 3rd due to 2nd vertex above is "new"—we'll see that the 3rd diagram is required for consistency, as expected from gauge invariance of interaction terms: without \(A \mu \phi^+ \phi\), theory is not gauge-invariant):

\[ i \mathcal{M} = (ie)^2 i \left[ \frac{(p+p-k')^\mu \epsilon_{\gamma'}^{\star \mu}(k') \left[ (p+p'-k')^\nu \epsilon_{\nu}(k) \right]}{\left[ (p-k')^2 - m^2 \right]} \right] \]

1st diagram

\(\mu\) scalar mass

2nd diagram + \[ \frac{\left[ (p+p+k)^\mu \epsilon_{\gamma'}^{\star \mu}(k') \left[ (2p+k)^\nu \epsilon_{\nu}(k) \right]}{(p+k)^2 - m^2} \]

3rd diagram
We would like to show $k^\mu M^\nu = 0$, where $M^\nu$ is above Feynman amplitude, but without $E_{r\nu} (k)$. Use (i) $p^2 = p'^2 = m^2$ (on-shellness of external scalars), (ii) $k^2 = k'^2 = 0$ (on-shellness of external photons), (iii) $p + k = p' + k'$ (overall energy-momentum conservation) and (iv) $k \mu E_{r\nu} (k)$

$= k^\nu \epsilon^*_{r\nu} (k) = 0$ (Lorentz gauge)

$\eta_{\nu \nu} E_{\nu} \rightarrow k_{\nu}$

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$= \epsilon^*_{\nu} \epsilon_{r\nu} (k' = 0)$

$\epsilon_{r\mu} = \epsilon_{r\mu} (k')$

$\eta_{\nu \nu} E_{\nu} \rightarrow k_{\nu}$

$= - \epsilon^2 \left\{ (2p \cdot \epsilon^* \cdot (2p' \cdot k) \right\} / \left( -2p' \cdot k \right) +$

$+ (2 p \cdot k)(2 p' \cdot \epsilon^*) / (2 p \cdot k) \} + 2 \epsilon^2 k \cdot \epsilon^*$

$= 2 \epsilon^2 (p - p' + k) \cdot \epsilon^* = 2 \epsilon^2 k' \cdot \epsilon^* = 0$

$k \eta_{\nu \nu} E_{\nu} \rightarrow k_{\nu}$

use (iii) here

use (iv) here