1 Solution1

\[ |L\rangle = \frac{1}{\sqrt{2}}(|e_x > - i|e_y >) \]
\[ |R\rangle = \frac{1}{\sqrt{2}}(|e_x > + i|e_y >) \]
\[ P_L = |L><L| = \frac{1}{2}(|e_x > - i|e_y >)(|e_x > + i|e_y >) = \frac{1}{2}(|e_x > - i|e_y > - i|e_x > + i|e_y > + |e_x > + i|e_y > + |e_y > - e_y >) \]
If we set \( |e_x > = |\alpha >, |e_y > = |\beta > \), \( |e_x | = 1, |e_y | = 1 \)

\[ P_L = \frac{1}{2} \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \]
Similarly
\[ P_R = \frac{1}{\sqrt{2}}(|1, -i>) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \]
In the same basis,

\[ P_x = |e_x > < e_x| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]

\[ P_y = |e_y > < e_y| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]

Intensity after passing through X,R then Y polarizer should be
\[ |< e_y|P_y P_R P_X |e_x >|^2 = \frac{1}{4} \]

2 Solution2

(a). \( tr(XY) = \sum_i < i|XY|i = \sum_i < i|X|j>< j|Y|i = \sum_i < j|X|i>< i|Y|j = \sum_j < j|YX|i = tr(XY) \).

(b). \( < i|XY|j = < X^\dagger i|Y|j = < Y^\dagger X^\dagger i|j > \)

(c). \( A|i > = a_i|i > \) for all i, thus \( e^{i\hat{f}(\alpha)} = \sum_{n=0} |\hat{f}(\alpha)|^n > = \sum_{n=0} |\hat{f}(\alpha)|^n > \)

(d). \( \sum_a \psi_a(\vec{x})\psi_a^\dagger(\vec{x}''') = \sum_a < a|\vec{x}'''> < a''|\vec{x}'' > = \sum_a < \vec{x}''|a'' > < a'|a'' > = \sum_a < \vec{x}''|a'' > < a''|\vec{x}'' > = \rho_{a''} \rho_{d''} \)

3 Solution3

(a). \( |a'' > = \sum_{a',a''} |a''|a'' > < a'|a'' > < a''|a'' > = \sum_{a',a''} |a''|a'' > < a'|a'' > \times ( < a''|a'' > < a''|a'' > \}

Hence \( |a'' > = \sum_{a',a''} |a''|a'' > < a'|a'' > < a''|a'' > \}

(b). \( |a > = |S_z = h/2 > = |+ >, |\beta > = |S_z = h/2 > = |1/2(|+ > + |− >) \}

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so \( |\alpha><\beta| = \frac{1}{\sqrt{2}}(|+><+| + |+><-|) \)

if \(|+> = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |-> = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\), then \(|\alpha><\beta| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\).

4 \quad \textbf{Solution 4}

The orthonormality property is \(<+|+>=<|-->=1, <+|-->=<--|=0\). Hence using the explicit representations of \(S_i\) in terms of the linear combinations of bra-ket products: \(S_x = \frac{i}{2}(|+><-| + |-><+|), \quad S_y = \frac{i}{2}(-i|+><-| - |><+|), \quad S_z = \frac{h}{2}(|+><+| - |><-|)\), we obtain by elementary calculation \([S_i, S_j] = i\epsilon_{ijk}\hbar S_k\) and \(S_y, S_j = \hbar^2/2\delta_{ij}\).

5 \quad \textbf{Solution 5}

Rewrite \(H = \frac{1}{2}(H_{11} + H_{22})(|1><1| + |2><2|) + \frac{1}{2}(H_{11} - H_{22})(|1><-1| - |2><-2|) + H_{12}(|1><2| + |2><1|)\), where the three operator terms on r.h.s behave like \(I, S_z, S_y\) respectively. Because the identity operator \(I\) remains the same under any change of basis, we ignore the \(\frac{1}{2}(H_{11} + H_{22})(|1><1| + |2><2|)\) for moment. Compare now with the spin \(\frac{1}{2}\) problem \(\hat{S} \cdot \hat{n} = \frac{\hbar}{2}n_x(|+><-| + |-><+|)\). The analogy is \(\frac{h}{2}n_x\rightarrow H_{12}, \frac{h}{2}n_y \rightarrow 0, \frac{h}{2}n_z \rightarrow \frac{1}{2}(H_{11} - H_{22})\). So one of the energy eigenkets is \(\cos(\beta/2)|1> + \sin(\beta/2)|2>\) where \(\beta\), analogous to \(\tan^{-1}(n_x/n_z)\), is given by \(\beta = \tan^{-1}\left(\frac{2H_{12}/\hbar}{H_{11}-H_{22}}\right)\). The other energy eigenket can be written down by the orthogonality requirement (or by letting \(\beta \rightarrow \beta\pi\)) as \(-\sin(\beta/2)|1> + \cos(\beta/2)|2>\). The energy eigenvalues can be obtained by diagonalizing

\[
\begin{pmatrix}
\frac{1}{2}(H_{11} - H_{22}) & H_{12} \\
H_{12} & \frac{1}{2}(H_{11} - H_{22})
\end{pmatrix}
\]

But they can also be obtained by comparing with the spin \(\frac{1}{2}\) problem:

\[
\left(\frac{\hbar}{2}n_x\right)^2 + \left(\frac{\hbar}{2}n_z\right)^2 = \frac{\hbar^2}{4} \rightarrow \text{eigenvalue} \frac{\hbar}{2}
\]

so by analogy the eigenvalue in our case is \([\frac{1}{4}(H_{11} - H_{22})^2 + H_{12}^2]^{1/2}\). We must still add to this the \(\frac{1}{2}(H_{11} + H_{22})\). The final answer is \(\frac{1}{2}(H_{11} + H_{22}) \pm [\frac{1}{4}(H_{11} - H_{22})^2 + H_{12}^2]^{1/2}\), where \(\pm\) is the analogue of parallel (anti-parallel) spin direction to \(\hat{n}\). For \(H_{12} = 0\), we get \(\beta = 0\) or \(\pi\). The eigenvalues are just \(H_{11}, H_{22}\).
6 Solution6

Here $\vec{S} \cdot \hat{n} |\hat{n}; + > = \frac{\hbar}{2} |\hat{n}; + >$ and $|\hat{n}; + > = \cos(\gamma/2)|+ > + \sin(\gamma/2)|- > = \begin{pmatrix} \cos(\gamma/2) \\ \sin(\gamma/2) \end{pmatrix}$. It is easily seen that the eigenket of $S_x$ belonging to eigenvalue $+\hbar/2$ is $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus (a) the probability of getting $+\hbar/2$ when $S_x$ is measured is $\left| \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right|^2 = \frac{\hbar}{2} \cos(\gamma/2), \sin(\gamma/2)$. (b) $\langle S_x \rangle = \frac{\hbar}{2} \begin{pmatrix} \cos(\gamma/2), \sin(\gamma/2) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \cos(\gamma/2) \\ \sin(\gamma/2) \end{pmatrix} = \frac{\hbar}{2} \sin(\gamma)$. Hence $\langle (S_x - \langle S_x \rangle)^2 \rangle = \langle S_x^2 \rangle - \langle S_x \rangle^2 = \hbar^2/4 - (\hbar^2/4)\cos^2(\gamma)$. Answers are entirely reasonable for $\gamma = 0, \pi$ (parallel and anti-parallel to OZ), and for $\gamma = \pi/2$ (along OX).