Lecture 9. Momentum Representation, Change Basis, More Examples, Wednesday, Sept. 21

Work out the momentum operator in the $x$-representation following the textbook.

The eigenvalues of $\hat{p}$ are also continuous and span a one-dimensional real axis. Eigenstates $|p\rangle$ can be chosen as a basis in the Hilbert space,

$$\langle p| p' \rangle = \lambda \delta(p - p') ,$$

$$\int \frac{dp}{\lambda} |p \rangle \langle p| = 1 ,$$

where $\lambda$ can be chosen as anything. It is 1 in JJS; I usually choose $2\pi\hbar$.

The momentum eigenstates can be expressed in $|x\rangle$ basis. The eigen-equation in $x$-rep is well-known,

$$-i\hbar \frac{d}{dx} \langle x|p \rangle = p \langle x|p \rangle$$

and the solution is

$$\langle x|p \rangle = e^{ipx/\hbar} .$$

We note that

$$\int_{-\infty}^{\infty} e^{ikx} dx = 2\pi \delta(x) .$$

Momentum representation: choose $|p\rangle$ as a basis, we have the momentum space wave function

$$\psi(p) = \langle p|\psi \rangle .$$

It can be shown that the momentum space wave function is related to the coordinate space wave function by simple Fourier transformation.

The position operator in the momentum representation: $\hat{x}$ is the generator of translation in the momentum space,

$$\hat{x} = i\hbar \frac{d}{dp} .$$

Gaussian wave packet.

$$\langle x|\alpha \rangle = \frac{1}{\pi^{1/4}\sqrt{d}} \exp\left(ikx - x^2/2d^2\right)$$
Plot its probability density. Calculate the expectation value $\langle \alpha | x | \alpha \rangle = 0$ because of the symmetry. On the other hand $\langle \alpha | x^2 | \alpha \rangle = d^2/2$, so one has, $\Delta x = d/\sqrt{2}$. Likewise, $\Delta p = \hbar/\sqrt{2}d$. Therefore, $\Delta x \Delta p = \hbar/2$.

Review harmonic oscillator, one-dimensional square well potential problems.

**Change of Basis:** Consider two bases, $|i\rangle$ and $|j\rangle$. The two bases are related by a unitary transformation $|i'\rangle = U|i\rangle$

(92)

where $U^\dagger U = U U^\dagger = 1$. If we insert $\sum_j |j\rangle \langle j| = 1$, then we have,

$$|i'\rangle = \sum_j |j\rangle \langle j| U|i\rangle = \sum_j |j\rangle U_{ji}$$

(93)

where $U_{ij} = \langle i| U|j\rangle$. We can write the above equation in a matrix form,

$$\begin{pmatrix} |1\rangle', |2\rangle', \ldots \end{pmatrix} = \begin{pmatrix} |1\rangle, |2\rangle, \ldots \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} & \ldots \\ U_{21} & U_{22} & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

(94)

where the basis vectors appear as a row matrix.

Suppose a vector $|\psi\rangle = \sum_i c_i |i\rangle$ in the old basis, and we can represent $|\psi\rangle$ as a column matrix of $c_i$, or

$$|\psi\rangle = \begin{pmatrix} |1\rangle, |2\rangle, \ldots \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix}$$

(95)

The same vector can be expressed as $|\psi\rangle = \sum_i c'_i |i'\rangle$ in the new basis. Then it is easy to see that

$$c'_i = \sum_j U^{-1}_{ij} c_j$$

(96)

or we can replace the $U^{-1}$ by $U^\dagger$ because of the unitarity.

**In some textbooks, the $U$ here is denoted by $U^{-1}$.

Consider also an operator $O = \sum_{ij} O_{ij} |i\rangle \langle j|$, which can be written also as a matrix form,

$$O = \begin{pmatrix} |1\rangle, |2\rangle, \ldots \end{pmatrix} \begin{pmatrix} O_{11} & O_{12} & \ldots \\ O_{21} & O_{22} & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} |1\rangle \\ |2\rangle \end{pmatrix}$$

(97)
From this, it is easy to show that

\[ O'_{ij} = \sum_{kl} U^\dagger_{ik} O_{kl} U_{lj} \]  

or simply \( O' = U^\dagger O U \) in the matrix sense.

The trace of an matrix is independent of bases.

Suppose we have a matrix \( O \), and we diagonalize it in the old basis \( |i\rangle \). Suppose all eigenvectors are \( |\lambda_n\rangle \). Then in the \( |\lambda_n\rangle \) basis, the matrix is diagonal with eigenvalue \( \lambda_n \). The transformation matrix from the old to the new basis is \( |\lambda_i\rangle = U|i\rangle \). Thus we can write,

\[ O = U^\dagger O_D U \]

where \( O_D \) is the diagonal matrix, and \( U \) is a matrix whose columns are formed by eigenvectors.

Example of \( \sigma_y \).

If two observables are related by unitary transformation \( A, B = UAU^{-1} \), we say \( A \) and \( B \) are unitary equivalent observables. It is easy to see that \( A \) and \( B \) have exactly the same eigenvalues, and their eigenstates are unitary transformation of each other. \( S_x \) and \( S_y \) and \( S_z \) are unitary equivalent observables.

Discuss homework problems. Solve new problems, time permits.

Hints: 1.21: Start with the normalized wave function in the square well potential.

\[ \psi_n = \sqrt{\frac{2}{a}} \sin \frac{n\pi}{a} \]

where \( n = 1, 2, ..., \). In the x-representation, \( p = -i\hbar d/dx \).

1.26: Consider \( S_x \) as the old basis. Diagonalize \( S_x \) in the old basis. Express \( U \) in the matrix form.

1.29: Consider 1D case, multi-D is easy because \( [x_i, x_j] = 0 \). Show it is true for a monomial \( p^n \). Classical Poisson bracket definition (1.6.48).

1.33: a) Insert \( \int dx |x\rangle \langle x| = 1 \) and use \( ixe^{ip} = d(e^{ixp})/dp \) b) Take the matrix element of the exponential operator between \( \langle x| \) and \( |p\rangle \), and see what you get!