Translation and momentum operator

In classical physics, it is known that if there is a continuous symmetry in the lagrangian of a system, there is a corresponding conservation law (see Landau & Lifshitz, Mechanics). For instance, if the lagrangian is independent of the location of the system (or translational invariant), linear momentum is conserved, and if it is independent of the orientation of the system in space, angular momentum is conserved. In quantum mechanics, the story is similar. Here we first consider translation.

Suppose we start with a state which is localized at \( x \), with ket \( |x\rangle \). Suppose we translate this state to \( x + dx \), with ket \( |x + dx\rangle \). \([\text{This is the so-called active point of view in the sense that one translates the physical system, rather than change the coordinates.}]\) This can be done with an operator in the Hilbert space

\[
T(dx)|x\rangle = |x + dx\rangle. \quad (71)
\]

Likewise, if the state is \( |\psi\rangle \), the translated state is \( |\psi'\rangle = T(dx)|\psi\rangle \). Intuitively, it is easy to see that the wave function of a translated state \( \psi'(x) \) is obtained from the original by \( \psi'(x) = \psi(x - dx) \). Thus \( \langle x|T(dx) = \langle x - dx| \), or \( T^\dagger(dx)|x\rangle = |x - dx\rangle \).

The translation operator must satisfy the following properties,

a) Unitary: \( T^\dagger T = TT^\dagger = 1 \).

b) Identity: \( \lim_{dx \to 0} T(dx) = 1 \).

c) Multiplication: \( T(dx)T(dx') = T(dx + dx') \).

The above indicates that all translations form a continuous group \( U(1) \) [unitary group which depends on a single parameter].

We can write \( T(dx) \) as

\[
T(dx) = 1 - iKdx \; , \quad (72)
\]

where \( K \) is a hermitian operator (which is also called the generator of translation). It is easy to show the following commutation relation,

\[
[x, T(dx)] = dx \; . \quad (73)
\]

Or more simply (for 3D case)

\[
[x_i, K_j] = i\delta_{ij} \; , \quad (74)
\]

22
i.e., the generator of translation does not commute with the coordinate.

If a system is invariant under translation, the Hamiltonian operator of the system is invariant under the above transformation. \( T(dx)HT^{-1}(dx) = H \).

This means the hamiltonian commutes with \( K \)

\[
[H, K] = 0 .
\] (75)

Hence \( H \) and \( K \) can be diagonalized simultaneously. Therefore, if a system has definite energy, it can also have definite eigenvalue of \( K \). Therefore \( K \) is a conserved quantity.

Following classical mechanics, we can identify \( \vec{K} \) as the linear momentum

\[
\vec{K} = \vec{p}/\hbar ,
\] (76)

where \( \hbar \) is a constant of proportionality and has the dimension of action. And then the commutation relation becomes,

\[
[x_i, p_j] = i\hbar \delta_{ij} .
\] (77)

For this we know that \( x \) and \( p \) are incompatible and obey the following commutation relation

\[
\Delta x \Delta p \geq \hbar/2 .
\] (78)

Therefore the momentum and coordinate cannot be measured simultaneously (Heisenberg).

Thus we have recovered all independent commutation relations among coordinates and momentum.

It is easy to show that a finite translation \( a \) can be written as

\[
T(a) = e^{-ipa/\hbar} .
\] (79)

We call the momentum as the generator of transformation.

Since translations in different spatial directions are commutable, \( T(dx)T(dy) = T(dy)T(dx) \), we have

\[
[p_i, p_j] = 0 .
\] (80)

What does the momentum operator look like in the x-representation? Well, we know that \( \psi'(x) = \psi(x - a) \), or

\[
\psi'(x) = \exp(-a \frac{d}{dx} )\psi(x)
\] (81)
or \( \langle x | T(dx) | \psi \rangle = \psi(x) - a \frac{d}{dx} \langle x | \psi \rangle \) Therefore, we immediately have,

\[
\langle x | \hat{p} | x' \rangle = \frac{\hbar}{i} \frac{d}{dx} \delta(x - x') . \tag{82}
\]

The momentum operator is a local operator! Notice that the delta function derivative with respect to \( x' \) differs from that with respect to \( x \) by a minus sign! Since the delta function always cancels an integration, we can write the \( p \) operator just as the differential operator, omitting at the same time the integral, so,

\[
P | \psi \rangle \rightarrow \frac{\hbar}{i} \frac{d}{dx} \psi(x) \tag{83}
\]

Likewise, one can show that

\[
\langle \beta | p | \alpha \rangle = \int dx \psi^*_\beta(x) \left( \frac{\hbar}{i} \frac{d}{dx} \right) \psi_\alpha(x) \tag{84}
\]

We can also add any number of power on the \( p \) operator and get a similar result.