Solution to PHYS606 Problem Set 3

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1 Problem 1: Angular momentum

1.1 a)

Using

\[ \partial_{\mu}T^{\mu\nu} = -\frac{1}{c}F^{\nu\lambda}J_{\lambda} \quad (1) \]

We have

\[ \partial_{\lambda}M^{\mu\nu\lambda} = \partial_{\lambda}(x^{\mu}T^{\nu\lambda} - x^{\nu}T^{\mu\lambda}) \]

\[ = \delta_{\lambda}^{\mu}T^{\nu\lambda} - \delta_{\lambda}^{\nu}T^{\mu\lambda} + x^{\mu}\partial_{\lambda}T^{\nu\lambda} - x^{\nu}\partial_{\lambda}T^{\mu\lambda} \]

\[ = x^{\mu}\partial_{\lambda}T^{\nu\lambda} - x^{\nu}\partial_{\lambda}T^{\mu\lambda} \]

\[ = \frac{1}{c}(x^{\nu}F^{\mu\alpha}J_{\alpha} - x^{\mu}F^{\nu\rho}J_{\rho}) \]

\[ = \frac{1}{c}(x^{\mu}J_{\alpha}F^{\alpha\nu} - x^{\nu}J_{\alpha}F^{\alpha\mu}) \]

1.2 b)

The conserved “current” corresponding to rotation is \( M^{\mu\nu\lambda} \), which means that

\[ \partial_{\lambda}M^{\mu\nu\lambda} = 0 \quad (2) \]

in the absence of external currents that are coupled to the field.

In general, for a conserved “current” \( j^{\nu} \) corresponding to a continuous symmetry of a system, one has

\[ \partial_{0} \int d^{3}x j^{0} = - \int d^{3}x \partial_{i}j^{i} = - \int dS \cdot \vec{j} = 0 \quad (3) \]

\(^{1}\)This set of solutions is partly adapted from Vivek Saxena’s.
if we set the region of integration to be large enough so that the surface term vanishes. 
This means that

\[ Q^0 = \int d^3x j^0 \]  

(4)

is a conserved quantity, usually termed "conserved charge". Specifically for \( M^{\mu\nu} \), we have

\[ Q^\nu_M = \int d^3x M^{\mu\nu} \]  

(5)

are the conserved charges corresponding to rotation. Using the expression of \( T^{\mu\nu} \)

\[ T^{\mu\nu} = \begin{pmatrix}
\mathcal{E} & \frac{1}{c} S_x & \frac{1}{c} S_y & \frac{1}{c} S_z \\
\frac{1}{c} S_x & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\
\frac{1}{c} S_y & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\
\frac{1}{c} S_z & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz}
\end{pmatrix} \]  

(6)

in which

\[ \mathcal{E} = \frac{1}{8\pi} (|\vec{E}|^2 + |\vec{B}|^2) \]  

(7)

\[ \vec{S} = \frac{c}{4\pi} \vec{E} \times \vec{B} \]  

(8)

Therefore

\[ M^{ij0} = x^j T^{i0} - x^i T^{j0} = \frac{1}{c} (x^j S^i - S^i x^j) = \frac{1}{c} \epsilon^{ijk} (\vec{x} \times \vec{S})_k \]  

(9)

The corresponding conserved charge is

\[ Q^i_M = \int d^3x \frac{1}{c} \epsilon^{ijk} (\vec{x} \times \vec{S})_k \]  

(10)

\[ M^{0i0} = x^i T^{00} - x^0 T^{i0} = x^i \mathcal{E} - x^0 \frac{1}{c} S^i = (\vec{x} \mathcal{E} - \frac{1}{c} \vec{S})^i \]  

(11)

The corresponding conserved charge is

\[ Q^{0i}_M = \int d^3x (\vec{x} \mathcal{E} - \frac{1}{c} \vec{S})^i \]  

(12)
2 Problem 2: Stress tensor in action

2.1 a)
Assuming that the two conducting planes are located at planes \( x = 0 \) and \( x = 0 \), then the electric field between them is
\[
\vec{E} = -4\pi \sigma \epsilon_x (0 < x < a)
\]
(13)

2.2 b)
The potential is
\[
\varphi(x) = -4\pi \sigma x (0 < x < a)
\]
(14)

2.3 c)
The electric field generated by the charged planes in the left is \( 2\pi \sigma \epsilon_x \), therefore the force (per unit area) it exerts on the other plane is
\[
\vec{f} = -2\pi \sigma^2 \epsilon_x
\]
(15)

2.4 d)
The stress tensor is the spatial part of stress-energy tensor
\[
T^{ij} = \begin{pmatrix}
-\frac{1}{\pi} |\vec{E}|^2 & 0 & 0 \\
0 & \frac{1}{\pi} |\vec{E}|^2 & 0 \\
0 & 0 & \frac{1}{\pi} |\vec{E}|^2
\end{pmatrix} = \begin{pmatrix}
-2\pi \sigma^2 & 0 & 0 \\
0 & 2\pi \sigma^2 & 0 \\
0 & 0 & 2\pi \sigma^2
\end{pmatrix}
\]
(16)

2.5 e)
And integrating over a surface (say \( x = a/2 \)) with area \( A \). Assume that the normal vector of the plane is in positive x direction, then one has
\[
\vec{f} = \frac{f^j}{A} \cdot \vec{e}_i = \int dA \hat{e}^j T^{ij} \cdot \vec{e}_i = -2\pi \sigma^2 \epsilon_x
\]
(17)
which shows that the force is attractive.

3 Problem 3: Stress tensor in action again

3.1 a)
Assuming that the two charges are located at \((0, L/2, 0)\) and \((0, -L/2, 0)\), then the electric field at reference point \((x, 0, 0)\) is
\[ \vec{E}(x) = \frac{2q|x|}{(x^2 + \frac{L^2}{4})^{3/2}} c_x \]  

(18)

3.2  b)

On the plane equidistant to the two charges, at reference point \((x, 0, 0)\)  the stress tensor is

\[
T^{ij} = \begin{pmatrix}
-\frac{1}{8\pi} |\vec{E}|^2 & 0 & 0 \\
0 & \frac{1}{8\pi} |\vec{E}|^2 & 0 \\
0 & 0 & \frac{1}{8\pi} |\vec{E}|^2
\end{pmatrix} = \begin{pmatrix}
-\frac{1}{8\pi} \cdot \frac{4q^2 x^2}{(x^2 + \frac{L^2}{4})^3} & 0 & 0 \\
0 & \frac{1}{8\pi} \cdot \frac{4q^2 x^2}{(x^2 + \frac{L^2}{4})^3} & 0 \\
0 & 0 & \frac{1}{8\pi} \cdot \frac{4q^2 x^2}{(x^2 + \frac{L^2}{4})^3}
\end{pmatrix}
\]

(19)

3.3  c)

To calculate the force one integrates the stress tensor over the equidistant plane (assuming that the normal vector of the plane is in the positive y direction)

\[
\vec{f} = f^i \hat{e}_i = (\int dA T^{ij}) \hat{e}_i = c_y \int_0^{\infty} \frac{1}{8\pi} \frac{4q^2 r^2}{(r^2 + \frac{L^2}{4})^3} 2\pi rdr = c_y \frac{q^2}{2} \int_0^{\infty} \frac{1}{(r^2 + \frac{L^2}{4})^3} dr = \frac{q^2}{L^2} c_y
\]

which shows that the force is repulsive between the two charges.

4  Problem 4: An action for Schrödinger’s equation

\[
S = \int dt d^3r \left( i\partial_t \psi + \frac{\hbar^2\nabla^2}{2m} - V \right) \psi = \int dt d^3r (\psi^* i\partial_t \psi - \frac{\hbar^2}{m} (\nabla \psi^*) (\nabla \psi) - V \psi^* \psi) = \int dt d^3r \hat{L}
\]

By

\[
\frac{\delta \hat{L}}{\delta \psi^*} = i\partial_t \psi - V \psi
\]  

(22)

\[
\frac{\delta \hat{L}}{\delta (\partial_t \psi^*)} = 0
\]  

(23)

\[
\partial_t \frac{\delta \hat{L}}{\delta (\partial_t \psi^*)} = -\frac{\hbar^2}{m^2} \nabla^2 \psi
\]  

(24)

Applying Euler-Lagrange equation with respect to \(\psi^*\) one has
\[
\frac{\delta \mathcal{L}}{\delta \psi^*} = \partial_i \frac{\delta \mathcal{L}}{\delta (\partial_i \psi^*)}
\]
\[
i \partial_t \psi = -\frac{\hbar^2 \nabla^2}{m^2} \psi + V \psi
\]  

5 Problem 5: Energy density of a straight wire

5.1 a)
By Ampere’s law we have
\[
\mathbf{B} = \frac{\mu_0 I}{2\pi r} \epsilon_\phi
\]  

5.2 b)
Energy density is simply
\[
\mathcal{E} = \frac{B^2}{2\mu_0} = \frac{\mu_0 I^2}{8\pi r^2}
\]  

5.3 c)
Energy per unit length is
\[
\mathcal{E}_l = \int_{\delta/2}^{R} \mathcal{E} \cdot 2\pi r dr
\]
\[
= \frac{\mu_0 I^2}{4\pi} \int_{\delta/2}^{R} \frac{dr}{r}
\]
\[
= \frac{\mu_0 I^2}{4\pi} \ln \left( \frac{2R}{\delta} \right)
\]  

5.4 d)
Assuming that R is 5 m, and the thickness is 10 mm, we have
\[
\mathcal{E}_l = \frac{\mu_0 I^2}{4\pi} \ln \left( \frac{2R}{\delta} \right)
\]
\[
= (10^{-7}) \cdot (20)^2 \cdot \ln \left( \frac{2 \cdot 5}{10 \times 10^{-3}} \right) J/m
\]
\[
= 2.7 \times 10^{-4} J/m
\]