Show that geodesics on spheres are great circles.

\[ ds^2 = a^2 d\theta^2 + a^2 \sin^2 \theta \, d\phi^2 \] so we must extremize

\[ J = a \int_0^{2\pi} f(\theta, \frac{d\theta}{d\phi}) \, d\phi \quad \text{with} \quad f(\theta, \frac{d\theta}{d\phi}) = \left[ \left( \frac{d\theta}{d\phi} \right)^2 + \sin^2 \theta \right]^{1/2} \]

\[ \frac{df}{d\theta} = \frac{\theta'}{\sqrt{\theta'^2 + \sin^2 \theta}} \]

\[ \frac{df}{d\theta} = \frac{\sin \theta \cos \theta}{\sqrt{\theta'^2 + \sin^2 \theta}} \quad \text{so the Euler-Lagrange eqn. is:} \]

\[ \frac{d}{d\phi} \frac{df}{d\theta'} - \frac{df}{d\theta} = 0 \quad \text{or} \]

\[ \frac{\partial^2 f}{\partial \theta \partial \theta'} \theta' \frac{\partial^2 f}{\partial \theta'^2} \theta'' - \frac{df}{d\theta} = 0. \]

\[ \frac{\partial^2 f}{\partial \theta^2} \theta'^2 + \frac{\partial^2 f}{\partial \theta'^2} \theta' \theta'' - \theta' \frac{df}{d\theta} = 0 \]

\[ \frac{d}{d\phi} \left[ \theta' \frac{df}{d\theta'} - f \right] = 0 \quad \text{or} \]

\[ \theta' \frac{df}{d\theta'} - f = \text{const} = -\frac{1}{k} \]

\[ \frac{\theta'^2}{\sqrt{\theta'^2 + \sin^2 \theta}} = -\frac{1}{k} \]