9. (15 pts) Consider the differential equation \( \dot{y} = y^2 + t \) with the initial condition \( y(0) = 1 \). This equation can be written in the form \( y = f(y, t) \) with \( f(y, t) = y^2 + t \).

(a) (3 pts) Do you expect a solution to exist and be unique in the vicinity of \( t = 0 \) and \( y = 1 \)? If so, why? **Yes:**

\[
\frac{\partial f}{\partial t} = 1; \quad \frac{\partial f}{\partial y} = 2y \quad \text{and} \quad \frac{\partial^2 f}{\partial y^2} = 2 \quad \Rightarrow
\]

Both \( \frac{\partial f}{\partial t} \) and \( \frac{\partial^2 f}{\partial y^2} \) are continuous near \( t=0 \) and \( y=1 \) (actually everywhere), and the criteria for the existence and uniqueness theorem are met.

(b) (12 pts) Using Euler’s crude method with a step size of \( h = .1 \), fill out the table below to find \( y \) at the time \( t = .2 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( t^n )</th>
<th>( y^n )</th>
<th>( f^n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>1.1</td>
<td>1.31</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>1.231</td>
<td>not needed</td>
</tr>
</tbody>
</table>

Euler says

\[ y^{n+1} = y^n + hf^n \]  
where  
\[ f^n = f(y^n, t^n) . \]

\( y^0 = 1 \) and \( t^0 = 0 \) \( \Rightarrow f^0 = 1^2 + 0 = 1 \); \( y' = y^0 + hf^0 = 1 + h = 1.1 \); \( f' = (y')^2 + t' = (1.1)^2 + .1 = 1.31 \); \( y^2 = y' + hf' \)

\[ = (1.1) + (.1)(1.31) = 1.1 + .131 = 1.231 \]