Goldstein 9.7

Part (a)

\( F_1(q, Q, t) \rightarrow F_2(q, P, t) \)

\[-P_i = \frac{\partial F_1}{\partial Q_i}\]  \hspace{1cm} (1)

\[F_2(q, P, t) = F_1(q, Q, t) + P_i Q_i\]  \hspace{1cm} (2)

\( F_1(q, Q, t) \rightarrow F_3(p, Q, t) \)

\[p_i = \frac{\partial F_1}{\partial q_i}\]  \hspace{1cm} (3)

\[F_3(p, Q, t) = F_1(q, Q, t) - p_i q_i\]  \hspace{1cm} (4)

\( F_1(q, Q, t) \rightarrow F_4(p, P, t) \)

\[p_i = \frac{\partial F_1}{\partial q_i}\]  \hspace{1cm} (5)

\[P_i = -\frac{\partial F_1}{\partial Q_i}\]  \hspace{1cm} (6)

\[F_4(p, P, t) = F_1(q, Q, t) - p_i q_i + P_i Q_i\]  \hspace{1cm} (7)

\( F_2(q, P, t) \rightarrow F_3(p, Q, t) \)

\[p_i = \frac{\partial F_2}{\partial q_i}\]  \hspace{1cm} (8)

\[Q_i = \frac{\partial F_2}{\partial P_i}\]  \hspace{1cm} (9)

\[F_3(p, Q, t) = F_2(q, P, t) - p_i q_i - P_i Q_i\]  \hspace{1cm} (10)

\( F_2(q, P, t) \rightarrow F_4(p, P, t) \)

\[p_i = \frac{\partial F_2}{\partial q_i}\]  \hspace{1cm} (11)

\[F_4(p, P, t) = F_2(q, P, t) - p_i q_i\]  \hspace{1cm} (12)
\[ F_3(p, Q, t) \rightarrow F_4(p, P, t) \]

\[ P_i = -\frac{\partial F_3}{\partial Q_i} \]  
\[ F_4(p, P, t) = F_3(p, Q, t) + P_i Q_i \]  

\textbf{Part (b)}

For an identity transformation, \( F_2 = q_i P_i \) and by equation (7), the type 4 generating function is

\[ F_4(p, P, t) = F_2(q, P, t) - p_i q_i \]
\[ = q_i P_i - p_i q_i \]
\[ = 0 \quad \text{as} \quad p_i = \frac{\partial F_2}{\partial q_i} = P_i \]  

For an exchange transformation, \( F_1 = q_i Q_i \) and by equation (4), the type 3 generating function is

\[ F_3(p, Q, t) = F_1(q, Q, t) - p_i q_i \]
\[ = q_i Q_i - p_i q_i \]
\[ = 0 \quad \text{as} \quad p_i = \frac{\partial F_1}{\partial q_i} = Q_i \]  

\textbf{Part (c)}

Consider a type 2 generating function \( F_2(q, P, t) \) of the old coordinates and the new momenta, of the form

\[ F_2(q, P, t) = f_i(q_1, \ldots, q_n; t) P_i - g(q_1, \ldots, q_n; t) \]  

where \( f_i \)'s are a set of independent functions, and \( g_i \)'s are differentiable functions of the old coordinates and time. The new coordinates \( Q_i \) are given by

\[ Q_i = \frac{\partial F_2}{\partial P_i} = f_i(q_1, \ldots, q_n; t) \]  

In particular, the function

\[ f_i(q_1, \ldots, q_n; t) = R_{ij} q_j \]  

where \( R_{ij} \) is the \( (i, j) \)-th element of a \( N \times N \) orthogonal matrix, generates an orthogonal transformation of the coordinates. Now,

\[ p_j = \frac{\partial F_2}{\partial q_j} = \frac{\partial f_i}{\partial q_j} P_i - \frac{\partial g}{\partial q_j} = R_{ij} P_i - \frac{\partial g}{\partial q_j} \]  

This equation can be written in matrix form, as

\[ p = \frac{\partial f}{\partial q} P - \frac{\partial g}{\partial q} \]
where \( p \) denotes the \( N \times 1 \) column vector \((p_1, \ldots, p_N)^T\), \( \partial g/\partial q \) denotes the \( N \times 1 \) column vector \((\partial g/\partial q_1, \ldots, \partial g/\partial q_n)^T\), and \( \frac{\partial f}{\partial q} \) denotes the \( N \times N \) matrix with entries

\[
\left( \frac{\partial f}{\partial q} \right)_{ij} = \frac{\partial f_i}{\partial q_j} = R_{ij}
\]  

From (22), the new momenta are given by

\[
P = \left( \frac{\partial f}{\partial q} \right)^{-1} \left( p + \frac{\partial g}{\partial q} \right)
\]  

\[
= R^{-1} \left( p + \frac{\partial g}{\partial q} \right)
\]  

\[
= R^{-1} (p + \nabla_q g)
\]  

As \( R \) is an orthogonal matrix, \( RR^T = R^T R = I \), so \( R^{-1} = R^T \) is also an orthogonal transformation.

This gives the required result: the new momenta are given by the orthogonal transformation \( (R^{-1}) \) of an \( n \)-dimensional vector \((p + \nabla_q g)\), whose components are the old momenta \((p)\) plus a gradient in configuration space \((\nabla_q g)\).

**Goldstein 9.25**

Part (a)

The given Hamiltonian is

\[
H = \frac{1}{2} \left( \frac{1}{q^2} + p^2 q^4 \right)
\]

The equation of motion for \( q \) is

\[
\dot{q} = \frac{\partial H}{\partial p} = pq^4
\]

Part (b)

Suppose we let \( Q^2 = 1/q^2 \) and \( P^2 = p^2 q^4 \). Then, \( Q = \pm 1/q \) and \( P = \pm pq^2 \). Now,

\[
\{Q, P\} = \{\pm 1/q, \pm pq^2\}
\]

\[
= \{q^{-1}, pq^2\}
\]

\[
= \{q^{-1}, p\} q^2 + p\{q^{-1}, q^2\}
\]

\[
= \left( \frac{\partial q^{-1}}{\partial q} \frac{\partial p}{\partial p} - \frac{\partial q^{-1}}{\partial p} \frac{\partial p}{\partial q} \right) q^2 + p \times 0
\]

\[
= \left( -\frac{1}{q^2} \right) q^2
\]

\[
= -1
\]
So, the signs on both $Q$ and $P$ cannot be identical. We take

$$Q = -\frac{1}{q}$$
$$P = pq^2$$

which is a valid canonical transformation. This gives the Hamiltonian,

$$H(Q, P) = \frac{1}{2}(P^2 + Q^2)$$

The equations of motion are

$$\dot{Q} = \frac{\partial H}{\partial P} = P$$
$$\dot{P} = -\frac{\partial H}{\partial Q} = -Q$$

So, $\ddot{Q} + Q = 0$, the solution to which is of the form $Q = A \cos t + B \sin t$. This gives $P = \dot{Q} = B \cos t - A \sin t$. Now,

$$q = -\frac{1}{Q} = -(A \cos t + B \sin t)^{-1}$$
$$p = PQ^2 = (B \cos t - A \sin t)(A \cos t + B \sin t)^2$$

so,

$$\dot{q} = (A \cos t + B \sin t)^{-2}(-A \sin t + B \cos t)$$

and hence

$$pq^4 = (B \cos t - A \sin t)(A \cos t + B \sin t)^2(A \cos t + B \sin t)^{-4} = (B \cos t - A \sin t)(A \cos t + B \sin t)^{-2} = \dot{q}$$

So, the solution to the transformed equation for $Q$ satisfies the original equation of motion for $q$.

**Problem 1**

**Part (a)**

$$\{X, P_x\} = \{x + \epsilon, p_x\} = \{x, p_x\} = 1$$

$$\{Y, P_y\} = \{y, p_x\} = 1$$
\[ \{Z, P_x\} = \{z, p_x\} \]
\[ = 1 \quad (40) \]
\[ \{X, P_y\} = \{X, P_x\} = \{Y, P_x\} = \{Z, P_x\} = \{Z, P_y\} = 0 \quad (41) \]
\[ \{X, X\} = \{Y, Y\} = \{Z, Z\} = \{P_x, P_x\} = \{P_y, P_x\} = \{P_z, P_z\} = \{P_y, P_y\} = \{P_z, P_x\} = 0 \quad (42) \]

So, this is a canonical transformation. It corresponds to a translated canonical coordinate system (translation along the \(x\)-direction in phase space).

\[ \frac{dX}{d\epsilon} = [X, P_x] = 1 \quad (43) \]

So \(P_x\) is the generator of the canonical transformation.

**Part (b)**

\[ \{X, P_z\} = \{x \cos \epsilon + y \sin \epsilon, p_x \cos \epsilon + p_y \sin \epsilon\} \]
\[ = \cos^2 \epsilon \{x, p_x\} + \sin^2 \epsilon \{y, p_y\} \]
\[ = 1 \quad (44) \]

\[ \{Y, P_y\} = \{-x \sin \epsilon + y \cos \epsilon, -p_x \sin \epsilon + p_y \cos \epsilon\} \]
\[ = \sin^2 \epsilon \{x, p_x\} + \cos^2 \epsilon \{y, p_y\} \]
\[ = 1 \quad (45) \]

\[ \{Z, P_z\} = \{z, p_z\} \]
\[ = 1 \quad (46) \]

Using properties of the Poisson Bracket, we also have

\[ \{X, P_y\} = \{X, P_x\} = \{Y, P_x\} = \{Z, P_x\} = \{Z, P_y\} = 0 \quad (47) \]
\[ \{X, X\} = \{Y, Y\} = \{Z, Z\} = \{P_x, P_x\} = \{P_y, P_z\} = \{P_z, P_z\} = \{P_y, P_z\} = \{P_z, P_x\} = 0 \quad (48) \]

So, this is a canonical transformation. It corresponds to a rotation about the \(z\)-axis in phase space.

\[ \frac{dX}{d\epsilon} = -x \sin \epsilon + y \cos \epsilon \quad (49) \]

whereas

\[ \{X, L_z\} = \{x \cos \epsilon + y \sin \epsilon, xp_y - yp_x\} = x \sin \epsilon - y \cos \epsilon \quad (50) \]

So,

\[ \frac{dX}{d\epsilon} = \{X, -L_z\} \quad (51) \]

Therefore, \(-L_z\) is the generator of the canonical transformation.
Part (c)

\[ \{X, P_x\} = \{x, p_x + \epsilon\} = 1 \]  \hspace{1cm} (52)

\[ \{Y, P_y\} = \{y, p_y\} = 1 \]  \hspace{1cm} (53)

\[ \{Z, P_z\} = \{z, p_z\} = 1 \]  \hspace{1cm} (54)

Using properties of the Poisson Bracket, we also have

\[ \{X, P_y\} = \{X, P_z\} = \{Y, P_x\} = \{Y, P_z\} = \{Z, P_x\} = \{Z, P_y\} = 0 \]  \hspace{1cm} (55)

\[ \{X, X\} = \{Y, Y\} = \{Z, Z\} = \{P_x, P_x\} = \{P_y, P_x\} = \{P_z, P_x\} = \{P_y, P_y\} = \{P_y, P_z\} = \{P_z, P_x\} = 0 \]  \hspace{1cm} (56)

So, this is a canonical transformation. It corresponds to a translation along the \( p_x \) direction in phase space. Now,

\[ \{P_x, -X\} = -\left( \frac{\partial P_x}{\partial q_i} \frac{\partial X}{\partial p_i} - \frac{\partial P_x}{\partial p_i} \frac{\partial X}{\partial q_i} \right) = 1 = \frac{dP_x}{d\epsilon} \]  \hspace{1cm} (57)

Therefore, \(-X\) is the generator of the canonical transformation.

Part (d)

\[ \{X, P_x\} = \{(1 + \epsilon)x, (1 + \epsilon)^{-1}p_x\} = 1 \]  \hspace{1cm} (58)

\[ \{Y, P_y\} = \{(1 + \epsilon)y, (1 + \epsilon)^{-1}p_y\} = 1 \]  \hspace{1cm} (59)

\[ \{Z, P_z\} = \{(1 + \epsilon)z, (1 + \epsilon)^{-1}p_z\} = 1 \]  \hspace{1cm} (60)

Using properties of the Poisson Bracket, we also have

\[ \{X, P_y\} = \{X, P_z\} = \{Y, P_x\} = \{Y, P_z\} = \{Z, P_x\} = \{Z, P_y\} = 0 \]  \hspace{1cm} (61)

\[ \{X, X\} = \{Y, Y\} = \{Z, Z\} = \{P_x, P_x\} = \{P_y, P_x\} = \{P_z, P_x\} = \{P_y, P_y\} = \{P_y, P_z\} = \{P_z, P_x\} = 0 \]  \hspace{1cm} (62)
So, this is a canonical transformation. It is a scaling transformation, which preserves the volume element in phase space. Suppose \( g \) is the generator of the scaling transformation. Then,

\[
\frac{\partial X}{\partial \epsilon} = x = [X, g] = [(1 + \epsilon) x, g]
\]

which implies

\[
\frac{x}{1 + \epsilon} = [x, g] = \frac{\partial x}{\partial x} \frac{\partial g}{\partial p_x} - \frac{\partial x}{\partial p_x} \frac{\partial g}{\partial x}
\]

that is,

\[
\frac{x}{1 + \epsilon} = \frac{\partial g}{\partial p_x}
\]

the solution to which is

\[
g = \frac{x p_x}{1 + \epsilon} + f(y, z, p_y, p_z)
\]

As \( dY/d\epsilon = y = [(1 + \epsilon) y, g] \) and \( dZ/d\epsilon = z = [(1 + \epsilon) z, g] \), following a similar argument for \( Y \) and \( Z \) (or by symmetry) we get

\[
g = \frac{x p_x}{1 + \epsilon} + \frac{y p_y}{1 + \epsilon} + \frac{z p_z}{1 + \epsilon} + \text{constant}
\]

as the generator of the scaling transformation.

**Problem 2**

As \( \eta \) is a canonical transformation, we have

\[
\frac{\partial \eta_i}{\partial \epsilon} = \{\eta_i, g\}
\]

So,

\[
\frac{\partial H}{\partial \epsilon} = \frac{\partial H}{\partial \eta_i} \frac{\partial \eta_i}{\partial \epsilon} = \frac{\partial H}{\partial \eta_i} \{\eta_i, g\} = \frac{\partial H}{\partial \eta_i} \frac{\partial \eta_i}{\partial \xi_j} J_{jk} \frac{\partial g}{\partial \xi_k} \quad \text{(as \( \xi \) is a canonical transformation)}
\]

\[
= \frac{\partial H}{\partial \eta_i} \frac{\partial \eta_i}{\partial \eta_j} J_{jk} \frac{\partial g}{\partial \eta_k} \quad \text{(as Poisson Brackets are invariant under canonical transformations)}
\]

\[
= \frac{\partial H}{\partial \eta_i} \delta_{ij} J_{jk} \frac{\partial g}{\partial \eta_k} = \frac{\partial H}{\partial \eta_i} J_{ij} \frac{\partial g}{\partial \eta_j} = \{H, g\} = -\dot{g}
\]

But since \( H \) is conserved, \( \frac{\partial H}{\partial \epsilon} = 0 \) and hence \( \dot{g} = 0 \). Therefore, \( g \) is conserved.
Problem 3

The quantity \( \Delta \), which was found to be an invariant of the system, can be expressed in terms of the canonical coordinates \( x, y, p_x, p_y \) as

\[
\Delta(x, y, p_x, p_y) = \frac{1}{2m}(p_x^2 - p_y^2) + \frac{1}{2}m\omega^2(x^2 - \alpha y^2) \tag{70}
\]

As \( \Delta \) is the conserved generator of a family of canonical transformations parametrized by an infinitesimal parameter \( \epsilon \), we must have

\[
\delta x = \epsilon \{ x, \Delta \} \tag{71}
\]
\[
\delta y = \epsilon \{ y, \Delta \} \tag{72}
\]
\[
\delta p_x = \epsilon \{ p_x, \Delta \} \tag{73}
\]
\[
\delta p_y = \epsilon \{ p_y, \Delta \} \tag{74}
\]

We consider each condition separately below.

\[
\delta x = \epsilon \{ x, \Delta \} = \epsilon \{ x, \frac{1}{2m}(p_x^2 - p_y^2) + \frac{1}{2}m\omega^2(x^2 - \alpha y^2) \} = \epsilon \{ x, \frac{p_x^2}{2m} \} = \frac{ep_x}{m} \tag{76}
\]
\[
\delta y = \epsilon \{ y, \Delta \} = \epsilon \{ y, \frac{1}{2m}(p_x^2 - p_y^2) + \frac{1}{2}m\omega^2(x^2 - \alpha y^2) \} = \epsilon \{ y, -\frac{p_y^2}{2m} \} = -\frac{ep_y}{m} \tag{77}
\]
\[
\delta p_x = \epsilon \{ p_x, \Delta \} = \epsilon \{ p_x, \frac{1}{2m}(p_x^2 - p_y^2) + \frac{1}{2}m\omega^2(x^2 - \alpha y^2) \} = \epsilon \{ p_x, \frac{1}{2}m\omega^2x^2 \} = -em\omega^2x \tag{78}
\]
\[
\delta p_y = \epsilon \{ p_y, \Delta \} = \epsilon \{ p_y, \frac{1}{2m}(p_x^2 - p_y^2) + \frac{1}{2}m\omega^2(x^2 - \alpha y^2) \} = \epsilon \{ p_y, -\frac{1}{2}m\omega^2\alpha y^2 \} = em\omega^2\alpha y \tag{79}
\]
Now, let $\epsilon = \delta \theta$ where $\theta$ is a parameter. Then, the above equations become

\[
\begin{align*}
\frac{dx}{d\theta} &= \frac{p_x}{m} \\
\frac{dy}{d\theta} &= -\frac{p_y}{m} \\
\frac{dp_x}{d\theta} &= -m\omega^2 x \\
\frac{dp_y}{d\theta} &= m\omega^2 y
\end{align*}
\]

So,

\[
\begin{align*}
\frac{d^2 x}{d\theta^2} + \omega^2 x &= 0 \\
\frac{d^2 y}{d\theta^2} + \omega^2 \alpha y &= 0
\end{align*}
\]

The solutions to which are

\[
\begin{align*}
x &= A \cos(\omega \theta) + B \sin(\omega \theta) \\
y &= C \cos(\omega \sqrt{\alpha} \theta) + D \sin(\omega \sqrt{\alpha} \theta)
\end{align*}
\]

and correspondingly

\[
\begin{align*}
p_x &= -m\omega A \sin(\omega \theta) + m\omega B \cos(\omega \theta) \\
p_y &= m\omega \sqrt{\alpha} C \sin(\omega \sqrt{\alpha} \theta) - m\omega \sqrt{\alpha} D \cos(\omega \sqrt{\alpha} \theta)
\end{align*}
\]

Using the subscript 0 to denote the “initial” coordinates and momenta, we have

\[
\begin{align*}
x_0 &= A \\
y_0 &= C \\
p_{x0} &= m\omega B \\
p_{y0} &= -m\omega \sqrt{\alpha} D
\end{align*}
\]

So,

\[
\begin{align*}
x &= x_0 \cos(\omega \theta) + \frac{p_{x0}}{m\omega} \sin(\omega \theta) \\
y &= y_0 \cos(\omega \sqrt{\alpha} \theta) - \frac{p_{y0}}{m\omega \sqrt{\alpha}} \sin(\omega \sqrt{\alpha} \theta) \\
p_x &= -m\omega x_0 \sin(\omega \theta) + p_{x0} \cos(\omega \theta) \\
p_y &= m\omega \sqrt{\alpha} y_0 \sin(\omega \sqrt{\alpha} \theta) + p_{y0} \cos(\omega \sqrt{\alpha} \theta)
\end{align*}
\]

Reverting to the notation in which $x_0, p_{x0}, y_0, p_{y0}$ denote the original coordinates and $X, Y, P_x, P_y$ denote the canonically transformed coordinates, we get the form of the canonical transformation as

\[
\begin{align*}
X &= x \cos(\omega \theta) + \frac{P_x}{m\omega} \sin(\omega \theta) \\
P_x &= p_x \cos(\omega \theta) - m\omega x \sin(\omega \theta) \\
Y &= y \cos(\omega \sqrt{\alpha} \theta) - \frac{P_y}{m\omega \sqrt{\alpha}} \sin(\omega \sqrt{\alpha} \theta) \\
P_y &= p_y \cos(\omega \sqrt{\alpha} \theta) + m\omega \sqrt{\alpha} y \sin(\omega \sqrt{\alpha} \theta)
\end{align*}
\]
where \( \theta \) is an arbitrary parameter, such that \( \theta = 0 \) corresponds to the untransformed coordinates. This canonical transformation is composed of two rotations in the 4-dimensional phase space (one involving \( X \) and \( P_x \) and the other involving \( Y \) and \( P_y \)), and its generator is the conserved quantity \( \Delta \).

**Problem 4**

**Part (a)**

\[ F_2(q, P, t) = \left( q + \frac{1}{2}gt^2 \right) \left( P - mgt \right) - \frac{P^2t}{2m} \quad (98) \]

Now,

\[ p = \frac{\partial F_2}{\partial q} = P - mgt \quad (99) \]

\[ Q = \frac{\partial F_2}{\partial P} = q + \frac{1}{2}gt^2 - \frac{Pt}{m} = q + \frac{1}{2}gt^2 - \frac{pt}{m} - gt^2 \quad (100) \]

So, the canonical transformation is

\[ P = p + mgt \quad (101) \]

\[ Q = q - \frac{pt}{m} - \frac{1}{2}gt^2 \quad (102) \]

**Part (b)**

\[ \{Q, Q\} = \{q - \frac{pt}{m} - \frac{1}{2}gt^2, q - \frac{pt}{m} - \frac{1}{2}gt^2\} = 0 \quad (103) \]

\[ \{P, P\} = \{p + mgt, p + mgt\} = 0 \quad (104) \]

\[ \{Q, P\} = \{q - \frac{pt}{m} - \frac{1}{2}gt^2, p + mgt\} \]

\[ = \frac{\partial Q}{\partial q} \frac{\partial P}{\partial P} - \frac{\partial Q}{\partial p} \frac{\partial P}{\partial q} \]

\[ = (1)(1) - \left( -\frac{t}{m} \right) (0) \]

\[ = 1 \quad (105) \]

So, the transformation satisfies the canonical Poisson Bracket relations.

**Part (c)**

The Lagrangian is

\[ L(q, \dot{q}) = \frac{1}{2}m\dot{q}^2 - mgq \quad (106) \]
The canonical momentum is

\[ p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \]  

(107)

So the Hamiltonian is

\[ H = p\dot{q} - L = \frac{p^2}{2m} + mgq \]  

(108)

Now, \( Q = q - \frac{pt}{m} - \frac{1}{2}gt^2 \), so

\[ \{Q, H\} = \left\{ q - \frac{pt}{m} - \frac{1}{2}gt^2, \frac{p^2}{2m} + mgq \right\} \]

\[ = \left\{ q, \frac{p^2}{2m} \right\} - \left\{ \frac{pt}{m}, mgq \right\} \]

\[ = \frac{1}{2m}\{q, p^2\} - gt\{p, q\} \]

\[ = \frac{p}{m} + gt \]  

(109)

Also

\[ \frac{\partial Q}{\partial t} = -\frac{p}{m} - gt \]  

(110)

So,

\[ \frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \{Q, H\} = 0 \]  

(111)

Also, \( P = p + mgt \), so

\[ \{P, H\} = \{ p + mgt, \frac{p^2}{2m} + mgq \} \]

\[ = mg\{p, q\} \]

\[ = -mg \]  

(112)

and

\[ \frac{\partial P}{\partial t} = mg \]  

(113)

So,

\[ \frac{dP}{dt} = \frac{\partial P}{\partial t} + \{P, H\} = 0 \]  

(114)
Part (d)

\[ \frac{\partial F_2}{\partial t} = gt(P - mgt) + \left( q + \frac{1}{2}gt^2 \right) (-mg) - \frac{P^2}{2m} \]

\[ = Pgt - \frac{3}{2}mg^2t^2 - mgq - \frac{P^2}{2m} \]

\[ = (p + mgt)gt - \frac{3}{2}mg^2t^2 - mgq - \frac{(p + mgt)^2}{2m} \]

\[ = -\frac{p^2}{2m} - mgq - mg^2t^2 \]

(115)

So, the Hamiltonian associated with Q, P is

\[ K = H + \frac{\partial F_2}{\partial t} \]

\[ = \frac{p^2}{2m} + mgq - \frac{p^2}{2m} - mgq - mg^2t^2 \]

\[ = -mg^2t^2 \]

(116)

So, the Hamiltonian \( K \) is zero up to time-dependent constant term \(-mg^2t^2\), but it is not a function of \( P \) and \( Q \) (which are constant with time, since \( \{Q, H\} = \{P, H\} = 0 \) as shown above).

Part (e)

\( Q \) and \( P \) are conserved quantities, that equal the initial position and the initial momentum respectively. They are constant with time, as \( q \) and \( p \) vary:

\[ q(t = 0) = Q \]
\[ p(t = 0) = P \]

Part (f)

\[ \frac{\partial F_2}{\partial q} = P - mgt = p \]

(117)

\[ \frac{\partial F_2}{\partial t} = Pgt - \frac{mg^2t^2}{2} - mgq - \frac{P^2}{2m} \]

(118)

\[ H = \frac{p^2}{2m} + mgq \]

\[ = \frac{1}{2m} \left( \frac{\partial F_2}{\partial q} \right)^2 + mgq \]

(119)

So,

\[ K = H + \frac{\partial F_2}{\partial t} = -mg^2t^2 \]

(as shown in part d)
implies
\[ H \left( q, \frac{\partial F_2}{\partial q} \right) + \frac{\partial F_2}{\partial t} = -mg^2 t^2 \] (120)

So, the Hamilton-Jacobi equation is satisfied, except for a time-dependent constant term appearing on the right hand side.

**Part (g)**

\[
f(Q, P, t) = F_2(q(Q, P, t), P, t) = \left( q + \frac{1}{2} gt^2 \right) (P - mgt) - P^2 t \frac{2}{2m}
\]
\[
= \left( Q + \frac{pt}{m} + gt^2 \right) (P - mgt) - P^2 t \frac{2}{2m}
\]
\[
= \left( Q + \frac{(P - mgt)t}{m} + gt^2 \right) (P - mgt) - P^2 t \frac{2}{2m}
\]
\[
= \left( Q + \frac{Pt}{m} \right) (P - mgt) - P^2 t \frac{2}{2m}
\]
\[
= QP + \frac{P^2 t}{2m} - Qmg - gP t^2
\] (121)

So,
\[
\frac{\partial f}{\partial t} = \frac{P^2}{2m} - mgQ - 2Pgt
\] (122)

Also, \( p = m\dot{q} = P - mgt \). So,
\[
L(q, \dot{q}) = \frac{p^2}{2m} - mgq
\]
\[
= \frac{1}{2m} (P - mgt)^2 - mg \left( Q + \frac{Pt}{m} - \frac{1}{2} gt^2 \right)
\]
\[
= \frac{1}{2m} (P^2 + m^2 g^2 t^2 - 2Pmgt) - mgQ - Pgt + \frac{1}{2} mg^2 t^2
\]
\[
= \frac{P^2}{2m} - 2Pgt - mgQ + mg^2 t^2
\]
\[
= \frac{\partial f(Q, P, t)}{\partial t} + mg^2 t^2
\] (123)

So, \( L(q(Q, P, t), \dot{q}(Q, P, t)) = \frac{\partial f(Q, P, t)}{\partial t} \) up to a time-dependent term \( mg^2 t^2 \).

**Problem 5**

The Hamilton-Jacobi equation, as expressed in the form
\[
H(q, \nabla S(q, P)) + \frac{\partial S(q, P)}{\partial t} = 0
\] (124)
was obtained by constructing a generating function of the form

\[ F = F_2(q, P, t) - Q_i P_i \]

where \( F_2 \) denotes a generic type-2 generating function. For such a choice of \( F \), the Hamiltonian \( \dot{K} = H + \frac{\partial F}{\partial t} \) is zero.

Now, consider a type-3 generating function \( F_3 \) of the old momenta and the new coordinates, such that the Hamiltonian \( K \) is zero. Therefore,

\[ \dot{Q}_i = \frac{\partial K}{\partial P_i} = 0 \]  \hspace{1cm} (125)

\[ \dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0 \]  \hspace{1cm} (126)

Now,

\[ q_i = -\frac{\partial F_3}{\partial p_i} = -(\nabla_p F_3)_i \]  \hspace{1cm} (127)

so,

\[ H(q(Q_p), p, t) + \frac{\partial F_3}{\partial t}(Q, p, t) = 0 \]  \hspace{1cm} (128)

where the old coordinates \( q \) have been expressed in terms of the old momenta and the new coordinates using equation (127). This is a PDE in \((n+1)\) variables \( p_1, \ldots, p_n, t \). Let \( \tilde{S} \) denote the solution of this PDE. Then, a solution of the form,

\[ F_3 \equiv \tilde{S} = \tilde{S}(p_1, \ldots, p_n; \alpha_1, \ldots, \alpha_{n+1}; t) \]  \hspace{1cm} (129)

where \( Q_i = \alpha_i \) are the constants of motion (for \( i = 1, \ldots, n \)), is consistent with equation (125). Here the constant \( \alpha_{n+1} \) must be a constant of integration, so the physically meaningful solution is of the form

\[ \tilde{S} = \tilde{S}(p_1, \ldots, p_n; \alpha_1, \ldots, \alpha_n; t) \]  \hspace{1cm} (130)

So, in terms of \( \tilde{S} \), equation (128) can be written as

\[ H(-\nabla_p \tilde{S}, p, t) + \frac{\partial \tilde{S}}{\partial t}(Q, p, t) = 0 \]  \hspace{1cm} (131)

which is of the desired form.