Problems

1. **SMM, Chapter 1, Problem 26.**

   When dealing with subatomic particles, it’s convenient to express their energy in electron volts (eV) \((p.36 \text{SMM})\). The conversion factor is \(1 \text{ eV} = 1.60 \times 10^{-19} \text{ J}\).

   The relativistic momentum is given by \(p = \gamma mu = \frac{mu}{\sqrt{1-\frac{u^2}{c^2}}}\).

   The mass of the proton is \(m_{\text{proton}} = 1.673 \times 10^{-27} \text{ kg}\).

   *Useful aid:* When converting momentum, which is normally expressed in units of \([\text{kg m/s}]\), into \([\text{MeV/c}]\) remember that the units of energy \([\text{Joule}]\) can also be expressed as \([\text{kg (m/s)}^2]\). Of course, \(c = 3 \times 10^8 \text{ m/s}\).

   Therefore directly plugging in \(m_{\text{proton}}\) and \(u\)...

   (a) for \(u = 0.010 \text{ c}\) → \(p = 5.019 \times 10^{-21} \text{ kg m/s} = 9.41 \text{ MeV/c}\)

   (b) for \(u = 0.500 \text{ c}\) → \(p = 2.898 \times 10^{-19} \text{ kg m/s} = 543 \text{ MeV/c}\)

   (c) for \(u = 0.900 \text{ c}\) → \(p = 1.036 \times 10^{-18} \text{ kg m/s} = 1943 \text{ MeV/c}\)

   (d) see (a) – (c)

2. **SMM, Chapter 1, Problem 27.**

   (a) Since the electron has a momentum that is 90% larger than the classical momentum, then this means that \(p = 1.90 p_{\text{classical}}\). Therefore,

   \[
   \frac{mu}{\sqrt{1-\frac{u^2}{c^2}}} = 1.90 mu
   \]

   The masses and the top velocity cancel. Solving for \(u\) we get,

   \[
   u = c \sqrt{1 - \frac{1}{(1.90)^2}} = 0.85 c
   \]

   (b) No change. The electron mass canceled out of the equation.

3. **SMM, Chapter 1, Problem 33.**

   A proton \((m_{\text{proton}} = 1.673 \times 10^{-27} \text{ kg})\) moves at \(v = 0.95 \text{c}\).

   (a) \(E_{\text{rest}} = m_{\text{proton}}c^2 = (1.673 \times 10^{-27})(3 \times 10^8)^2 = 1.5057 \times 10^{10} \text{ J} = 941 \text{ MeV}\).

   (b) \(E_{\text{total}} = \gamma m_{\text{proton}}c^2 = \frac{m_{\text{proton}}c^2}{\sqrt{1-\frac{v^2}{c^2}}} = 3.20 \text{ (941 MeV)} = 3013 \text{ MeV}\).
(c) $E_{\text{Kinetic}} = E_{\text{total}} - E_{\text{rest}} = 3.20 \ (941 \text{ MeV}) - (941 \text{ MeV}) = 2070 \text{ MeV}.$

4. SMM, Chapter 1, Problem 36.

There are various ways of doing this problem (all of them amount to the same thing): (1) Since we are given the kinetic energy, we can use \( \{ KE = \gamma mc^2 - mc^2 \} \) to find \( \gamma \) and then, the velocity \( v \). This solves part (b) first. The momentum is then given by \( \{ p = \gamma mv \} \). (2) Alternatively, we can compute the total energy using \( \{ E_{\text{total}} = \gamma mc^2 = KE + mc^2 \} \) and then use the Energy – Momentum relation \( \{ E_{\text{total}}^2 = p^2 c^2 + (mc^2)^2 \} \) to find the momentum \( p \). We then solve for the velocity using \( \{ p = \gamma mv \} \).

I will solve the problem using the first method. A proton has a rest mass energy: 
\[
E_{\text{rest}} = m_{\text{proton}} c^2 = (1.673 \times 10^{-27}) (3 \times 10^8)^2 = 1.5057 \times 10^{10} \text{ J} = 941 \text{ MeV}.
\]
Therefore, according to \( KE = \gamma mc^2 - mc^2 \),
\[
(50 \text{ GeV}) = \gamma (941 \text{ MeV}) - (941 \text{ MeV})
\]
\[
\rightarrow \gamma = 54.135...
\]
\[
\rightarrow v = 0.999829 \ c
\]

(a) The momentum is therefore,
\[
p_{\text{proton}} = \gamma m_{\text{proton}} v
\]
\[
= (54.135)(1.673 \times 10^{-27} \text{ kg})(0.999829 c)
\]
\[
= 2.72 \times 10^{-17} \text{ kg m/s} = 50.872 \text{ GeV/c}
\]

(b) As solved above, \( v = 0.999829 \ c \).

5. Follow up to Homework 2, problem 7. If we solve the scalar relativistic Newton’s equation, \( F = \frac{dp}{dt} = \frac{d}{dt} (\gamma mv) \), where \( \gamma \equiv \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \), for the velocity \( v(t) \), subject to the conditions of a constant force \( F_0 \) and initial velocity \( v_0 \), we get:
\[
v(t) = \left(\frac{F_0}{m} t + \frac{v_0}{c^2} \right) \left(1 + \frac{1}{c^2} \left(\frac{F_0}{m} t + \frac{v_0}{c^2}\right)^{-\frac{1}{2}}\right)^{-\frac{1}{2}}
\]

Consider a small 1 kg object starting from rest, subject to a force \( F_0 \) of 100 N.

(a) Using the non-relativistic speed equation (i.e. \( v(t) = \frac{F_0}{m} t + v_0 \)), how long would it take for the object to reach the speed of light? Give your answer in days.

(b) Using the relativistic speed equation, what’s the speed after the same period of time?

(c) Plot both equations as a function of time from 0 to the time computed in (a)?
Solution:

(a) Using \( v(t) = \frac{F_0}{m} t + v_0 \) and solving for \( t \) we get: \( t = \frac{m}{F_0} (v(t) - v_0) \). Plugging in \( F_0 = 100 \, \text{N}, m = 1 \, \text{kg}, v_0 = 0 \, \text{m/s} \) and \( v(t) = c \), we get \( t = 3 \times 10^6 \, \text{sec} \approx 34.7 \) days.

(b) Using the relativistic equation and plugging \( t = 3 \times 10^6 \, \text{sec} \) we get \( v(t) = 2.12 \times 10^8 \, \text{m/s} \).

(c) 

Extra Credit Problem 1. Derive the relativistic speed equation from problem 5. This looks a lot harder than it is. Give it a try.

This is a very similar problem to Homework 2, problem 7. The only difference is that we are not looking for \( x(t) \) but for \( v(t) \). The equations are more complicated but there are less steps. We start with \( F_0 = \frac{\partial}{\partial t} (\gamma m v) \) where \( \gamma \equiv \left( 1 - \frac{v^2}{c^2} \right)^{-\frac{1}{2}} \) and integrate both sides with respect to time.

\[
\int \frac{F_0}{m} \, dt = \int \frac{d}{dt} \left( \frac{v(t)}{\sqrt{1 - \frac{v^2(t)}{c^2}}} \right) dt
\]

\[
\frac{F_0}{m} t + c_1 = \frac{v(t)}{\sqrt{1 - \frac{v^2(t)}{c^2}}}
\]

Letting \( t = 0 \) and solving for \( c_1 \) we find \( c_1 = \frac{v_0}{\sqrt{1 - \frac{v_0^2}{c^2}}} \).

Substituting \( c_1 \) into the original equation,
\[
\frac{F_0}{m} t + \frac{v_0}{\sqrt{1 - \frac{v_0^2}{c^2}}} = \frac{v(t)}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

and solving for \(v(t)\) (after a bit of painful algebra) we find,

\[
v(t) = \left( \frac{F_0}{m} t + \frac{v_0}{\sqrt{1 - \frac{v_0^2}{c^2}}} \right) \left( 1 + \frac{1}{c^2} \left( \frac{F_0}{m} t + v_0 \right)^2 \right)^{1/2}
\]

7. Extra Credit Problem 2. Show that the relativistic speed equation reduces to the non-relativistic speed equation when \(v(t) \ll c\). Hint: To do this, we must assume that \(v_0 \ll c\) and that \(t\) is very small. Mathematically speaking, this is the equivalent of throwing out terms that have \(v_0^2/c^2\) and \(t^2\) in them.

We begin the solution by throwing out the obvious \(v_0^2/c^2\) terms directly. This is equivalent to expanding those terms using the binomial theorem and then throwing out the \(v_0^2/c^2\) terms. We are left with,

\[
v(t) \equiv \left( \frac{F_0}{m} t + v_0 \right) \left( 1 + \frac{1}{c^2} \left( \frac{F_0}{m} t + v_0 \right)^2 \right)^{1/2}
\]

Now let’s expand the squared term,

\[
v(t) \equiv \left( \frac{F_0}{m} t + v_0 \right) \left( 1 + \left( \frac{F_0}{m} t \right)^2 + \frac{2F_0 v_0 t}{mc^2} + \frac{v_0^2}{c^2} \right)^{1/2}
\]

Throwing out the \(t^2\) and \(v_0^2/c^2\) terms

\[
v(t) \equiv \left( \frac{F_0}{m} t + v_0 \right) \left( 1 + \frac{2F_0 v_0 t}{mc^2} \right)^{1/2}
\]

We use the binomial theorem since we are looking at \(t\) very small,

\[
v(t) \equiv \left( \frac{F_0}{m} t + v_0 \right) \left( 1 - \frac{F_0 v_0 t}{mc^2} \right)
\]

Expanding out the terms,

\[
v(t) \equiv \frac{F_0}{m} t - \left( \frac{F_0}{m} \right)^2 \frac{v_0}{c^2} + v_0 - \frac{F_0 v_0 t}{mc^2} \left( \frac{v_0}{c^2} \right)
\]

And throwing out the \(t^2\) and \(v_0^2/c^2\) terms one last time, we get the desired result:

\[
v(t) \equiv \frac{F_0}{m} t + v_0
\]