Exam 3 – Quantum Mechanics Formalism

Write your class ID number (NOT your name) on this exam.

Grading breakdown:

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Total points | 64 points
Problems

1. **(3 points)** What is the physical meaning of the normalization condition?

2. **(4 points)** Name two observables of a particle and state whether they are sharp or fuzzy.

3. **(3 points)** Explain the difference between phase and group velocities for a wave packet.

4. **(10 points; 2 each)** Of the following functions, which are candidates for the Schrödinger wave function of an actual physical system? For those that are not, state why they fail to qualify.

   - (a) 
   - (b) 
   - (c) 
   - (d) 
   - (e)
5. **(5 points)** Consider a particle in a box (a.k.a. the infinite square well). Calculate the probability that a particle, in the ground state, will be found in the middle half of the well (i.e. $\left[\frac{L}{4}, \frac{3L}{4}\right]$).

6. **(8 points; 2 each)** Graphically compare the 1\textsuperscript{st} and 2\textsuperscript{nd} bound state of the infinite and finite square well by carefully sketching their wave functions in the space provided.
7. **(5 points)** Consider an electron bound in a harmonic oscillator, 

\[ E_{\text{total}} = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2, \]

with an arbitrary uncertainty \( a (\Delta x \sim a) \). Using the uncertainty principle, (a) *estimate* the value of \( a \) that minimizes the total energy and (b) find this minimum energy and compare it to the exact harmonic oscillator ground state energy.

8. **(3 points)** The curve in the figure is alleged to be the plot of a computer-calculated wave function for the fifth energy level of a particle in the diagrammed one-dimensional potential well. Indicate the way in which the plot *fails* to be qualitatively correct.
9. (9 points; 3 each) Sketch careful, qualitatively accurate plots (on the space provided) for the 1st, 2nd, and 3rd bound state of a 1-D Coulomb potential. In your plots, don’t forget to clearly mark the location of the classical turning points. Notice that $V(r \leq 0) = \infty$. *Important: Check that your wave function has the correct symmetry, number of nodes, relative wavelengths, maximum values of amplitudes and relative rate of decrease outside the well.*
10. **(14 points)** The 2\textsuperscript{nd} bound state \((n = 1)\) wave function of the harmonic oscillator is given by

\[
\psi_1(x) = \left(\frac{1}{2a\sqrt{\pi}}\right)^{1/2} 2\left(\frac{x}{a}\right) e^{-\frac{x^2}{2a^2}} \text{ where } a = \sqrt{\frac{\hbar}{m\omega}}.
\]

(a) **(2 points)** Carefully sketch a qualitative but accurate plot of \(\psi_1(x)\). Clearly mark the location of the classical turning points.

(b) **(3 points)** Show that the wave function is normalized.

(c) **(6 points)** Compute the quantum uncertainty in the position, \(\Delta x\), in terms of \(a\).

(d) **(3 points)** What is \(\langle H \rangle\)?
Wave formalism
\[ \omega = 2\pi f \quad k = \frac{2\pi}{\lambda} \quad v_{\text{phase}} = \frac{\omega}{k} \quad v_{\text{group}} = \frac{d\omega}{dk} \mid_{k_0} \]
\[ v_{\text{group}} = v_{\text{phase}} \mid_{k_0} + k \frac{dv_{\text{phase}}}{dk} \mid_{k_0} \]
\[ \Delta x \Delta k \approx 1 \quad \Delta t \Delta \omega \approx 1 \]

Matter waves
\[ E = hf \quad \lambda = \frac{p}{k} \quad v_{\text{phase}} = c \sqrt{1 + \left(\frac{mc}{\hbar k}\right)^2} \]

Uncertainty principle
\[ \Delta x \Delta p_x \geq \frac{\hbar}{2} \quad \Delta E \Delta t \geq \frac{\hbar}{2} \]

Schrödinger equation
\[ -\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} + U(x)\psi(x) = E\psi(x) \]

Probabilities & Normalization
\[ P_{[a,b]} = \int_a^b \left| \psi(x,t) \right|^2 dx \quad \int_{-\infty}^{\infty} \left| \psi(x,t) \right|^2 dx = 1 \]

Particle in a box
\[ E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \quad \psi_n(x) = \frac{2}{L} \sin\left(\frac{n\pi x}{L}\right) \quad n = 1, 2, 3, \ldots \]

Harmonic oscillator
\[ [H_{\text{oscillator}}] = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2 \]
\[ E_n = \left(n + \frac{1}{2}\right) \hbar \omega \quad n = 0, 1, 2, \ldots \quad \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} \quad \text{(ground state)} \]

Expectation values
\[ \langle Q \rangle = \int_{-\infty}^{\infty} \psi^*(x)Q\psi(x)dx \quad \Delta Q = \sqrt{\langle Q^2 \rangle - \langle Q \rangle^2} \]

Operators
\[ \langle x \rangle = x \quad \langle p \rangle = \frac{\hbar}{i} \frac{\partial}{\partial x} \quad \langle H \rangle = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + U(x) \]

Integrals and Identities
\[ \int_{0}^{\infty} u^2 e^{-\beta u^2} \, du = \frac{1}{4\beta} \sqrt{\pi} \]
\[ \int_{0}^{\infty} u^4 e^{-\beta u^2} \, du = \frac{1}{2\beta^2} \sqrt{\pi} \]
\[ \sin 2\theta = 2\sin \theta \cos \theta \quad \sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta) \quad \cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta) \]
\[ \int_{0}^{\infty} u^3 e^{-\beta u^2} \, du = \frac{3}{8\beta^3} \sqrt{\pi} \]