Due Monday April 27

1) In class we argued that the minimizing the integral \( \int_0^t \left( \dot{x} - \frac{f(x)}{m} \right)^2 dt \) with respect to the parameters in a parameterized form of \( x(t) \) yields the “best” choice for \( x(t) \). Let us apply this to a problem where we know the answer, namely the harmonic oscillator: \( f(x) = -k x \) with boundary conditions \( x(0) = 0 \) \( \dot{x}(0) = v_0 \) and studied over 1/2 period from time \( 0 < t < \pi \omega \) where \( \omega = \sqrt{\frac{k}{m}} \). I will have you try various power laws. Hint the algebra is very nasty so you may want to use Mathematica.

a) First assume that \( x(t) = v_0 t + at^3 \) (note this form builds in the boundary conditions and the fact that the solution is an odd function of time. Find the best choice of \( a \).

b) First assume that \( x(t) = v_0 t + at^3 + bt^5 \) (note this form builds in the boundary conditions and the fact that the solution is an odd function of time. Find the best choice of \( a \) and \( b \)).

2) One way to test how well your solution works is compare your solution with the exact solution. Compare the solution to part a) and solution to part b) with the exact solution. You may take units where \( \omega = 1 \) and \( v_0 = 1 \).

3) Of course one already has an approximate solution for this problem in terms of a Taylor series which is of the same form as given by 1b): \( x(t) = v_0 (t - \frac{t^3}{6} + \frac{t^5}{120}) \). Compare the two approximation in 1b) to the Taylor series with the exact solution by plotting them. Which works better over the whole range? Why? Which works better at small \( t \)? Why?

4) Repeat the analysis in problems 1) and 2) for the case where the time is over a 1/4 period: \( 0 < t < \frac{\pi}{2 \omega} \). You should find far better agreement. Explain why. This illustrates a general point about the “richness” of the variational ansatz (“ansatz” is a fancy word for assumed form.)