Homework #2 Solutions

Question 1) Adiabatic Process for an ideal gas. We are given:

\[ \Delta U = Q + U \]
\[ U = \frac{f}{2} Nk_B T \]

Adiabatic Process means: \( Q = 0 \). Hence, from the 1st Law,

\[ \Delta U = W \]

And compressive work can be written as \( W = -P \Delta V \). With the above information:

\[ \frac{f}{2} Nk_B \Delta T = -P \Delta V \]

Now use the ideal gas Law \( PV = Nk_B T \) to eliminate Pressure \( P \),

\[ \frac{f}{2} Nk_B \Delta T = -\frac{Nk_B T}{V} \Delta V \]

Divide both sides by \( T \),

\[ \frac{f}{2} \frac{\Delta T}{T} = -\frac{\Delta V}{V} \]

Integrate both sides from the initial state \( (T_i, V_i) \) to the final state \( (T_f, V_f) \):

\[ \frac{f}{2} \int_{T_i}^{T_f} \frac{dT}{T} = -\int_{V_i}^{V_f} \frac{dV}{V} \]

\[ \frac{f}{2} \ln \frac{T_f}{T_i} = -\ln \frac{V_f}{V_i} \]

After exponentiating both sides of the equation,

\[ \left( \frac{T_f}{T_i} \right)^{\frac{f}{2}} = \frac{V_i}{V_f} \]

Multiply both sides by \( V_f T_f^{\frac{f}{2}} \), then

\[ V_f T_f^{\frac{f}{2}} = V_i T_i^{\frac{f}{2}} \]

So, in the other words,

\[ V T^{\frac{f}{2}} = C \]

where \( C \) is a constant. Now, we can use the ideal gas law to eliminate \( T \) in favor of \( P \):

\[ V \left( \frac{PV}{Nk_B} \right)^{\frac{f}{2}} = C \]
Both \(N\) and \(k_B\) are constant, so we can write
\[
P^\frac{f}{2}V^{\frac{1}{2}+1} = C'
\]
Where \(C'\) is another constant. Now raise both sides to the power \(2/f\) to get
\[
P\sqrt{\frac{2}{f}}V^{1+\frac{1}{2}} = C''
\]
and \(C''\) is also a constant.

Define \(\gamma \equiv \frac{L+2}{f}\), then we have
\[
PV^\gamma = \text{constant}
\]

**Discussion:** Alternative Method:

\[
dW = -PdV \\
dU = d\left(\frac{f}{2}NK_BT\right) = d\left(\frac{f}{2}PV\right) = \frac{f}{2}(VdP + PdV)
\]

*For an adiabatic process, \(Q = 0\), then*

\[
dU = dW \Rightarrow -PdV = \frac{f}{2}(VdP + PdV)
\]

*This can be written as*

\[
\gamma \frac{dV}{V} = -\frac{dP}{P}
\]

*Integrate both sides from the initial state \((V_i, P_i)\) to the final state \((V_f, P_f)\) to get:*

\[
PV^\gamma = \text{constant}
\]

**Question 2a)** We know that (from the above question), \(PV_i^\gamma = PV_f^\gamma\). Hence

\[
V_f = \left(\frac{P_i}{P_f}\right)^{\frac{1}{\gamma}}V_i
\]

where \(\gamma \equiv \frac{L+2}{f} = \frac{7}{5}\). Hence,

\[
V_f = (1\text{Liter})\left(\frac{1\text{atm}}{7\text{atm}}\right)^{\frac{7}{5}} = 0.25L
\]

**b)** The word done in compression is

\[
W = -\int_{V_i}^{V_f} PdV
\]

From the equation

\[
PV^\gamma = \text{constant}
\]

we can get

\[
P = \frac{C}{V^\gamma}
\]
Hence
\[ W = -C \int_{V_i}^{V_f} \frac{dV}{V^\gamma} = \frac{C}{\gamma - 1} [V_f^{1-\gamma} - V_i^{1-\gamma}] \]

Where \( C = P_fV_f^\gamma = P_iV_i^\gamma \), so that
\[ W = \frac{1}{\gamma - 1} [P_fV_f^{1-\gamma} - P_iV_i^{1-\gamma}] = \frac{1}{\gamma - 1} [P_fV_f - P_iV_i] \]

Before the calculation, we should change the units to SI,
\[ 1 \text{atm} = 1.013 \times 10^5 N/m^2 \]
\[ 1 \text{Liter} = 10^{-3} m^3 \]

Then we can get the result
\[ W = 189.9 N \cdot m = 189.9 J \]

**Discussion Alternative method:**
\[ W = \frac{f}{2} NK_B(T_f - T_i) = \frac{f}{2} (P_fV_f - P_iV_i) \]

c) We know from the last problem that \( VT_f^\frac{f}{2} \) = constant. Hence,
\[ V_fT_f^{\frac{f}{2}} = V_iT_i^{\frac{f}{2}} \]

So
\[ T_f = \left( \frac{V_i}{V_f} \right) ^{\frac{f}{2}} T_i = 300 K \left( \frac{1 L}{0.25 L} \right)^{2/5} = 522 K = 249^\circ C \]

which is pretty hot!

**Discussion Alternative methods: Method 1)** From the ideal gas law,
\[ \frac{T_f}{T_i} = \frac{P_fV_f}{P_iV_i} \approx \frac{7 \text{atm} \times 0.25 L}{1 \text{atm} \times 1 L} = \frac{7}{4} \]
\[ T_f \approx \frac{7}{4} \times 300 K = 525 K \]

**Method 2)** From the result of question 2b, (adiabatic process \( Q = 0 \))
\[ dU = W \]

you get the change of internal energy, then
\[ dU = \frac{f}{2} NK_B(T_f - T_i) \]
where $Nk_B = \frac{P1}{T}$.

**Question 3** Note that this probability is nothing but $g(N_h, N_t) / 2^N$ where $g(N_h, N_t) = \frac{N!}{N_h!N_t!}$ is the multiplicity function for spins that you calculated in class. I use $N_h$ to denote the number of heads and $N_t$ for the number of tails.

3a) $P(5, 5) = \frac{10!}{5!5!} = 0.246$ (i.e. 24.6%)

3b) $P(4, 6) = \frac{10!}{6!4!} = 0.205$ (i.e. 20.5%)

3c) The probabilities and the corresponding average is given in Table 1. Taking the square root of the average of $(\frac{N}{2} - N_h)^2$ we get $\delta N_{rms} = 1.58$ and $\delta N_{rms} / N = 0.158$.

**Discussion** How to get the root-mean squared deviation? The formula should take into consideration the probability of each deviation from the mean. And “⟨⟩” represents the average.

$$\delta N_{rms} = \langle(\frac{1}{2}N - N_h)^2\rangle^{1/2} = \left[ \sum_{N_h = 0}^{N_h = 10} (\frac{1}{2}N - N_h)^2 P(\frac{1}{2}N - N_h) \right]^{1/2}$$

Where $P(\frac{1}{2}N - N_h)$ is the probability of each $N_h$ as shown in the table above.

Let us compare this to the gaussian approximation. Remember that the spin excess 2s is equal to $N_h - N_t$ and the probability in terms of $s$ is given by $P(s) = \sqrt{\frac{2}{\pi N}} e^{-\frac{s^2}{2N}}$ and $\langle (\frac{N}{2} - N_h)^2 \rangle = \langle s^2 \rangle$, valid for large $N$ and $|s| << N$.

$$\langle (\frac{N}{2} - N_h)^2 \rangle = \langle s^2 \rangle = \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi N}} e^{-\frac{2s^2}{N}} s^2 ds = \frac{N}{4}$$

Hence $\delta N_{rms} = \sqrt{N}/2$ and $\delta N_{rms} / N = 1/(2\sqrt{N})$. Putting $N=10$ we get 0.158, which is very close to the exact solution. Remember for this problem $N$ is pretty small so $1/\sqrt{N}$ is not very small. In the next problem the approximation will work much better.

<table>
<thead>
<tr>
<th>$N_h$</th>
<th>Probability</th>
<th>$(\frac{N}{2} - N_h)^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.001</td>
<td>25</td>
</tr>
<tr>
<td>1</td>
<td>0.010</td>
<td>16</td>
</tr>
<tr>
<td>2</td>
<td>0.044</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>0.117</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>0.205</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>0.246</td>
<td>0</td>
</tr>
<tr>
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<td>0.205</td>
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<td>10</td>
<td>0.001</td>
<td>25</td>
</tr>
</tbody>
</table>

sum=1 average=2.508
**Question 4**) Again we will use the probability distribution function \( g(N_h, N_t) / 2^N \) as in question 3. However now the numbers are so big that we have to use the gaussian approximation (Try typing 5000! in your calculator!)

4a) \( P(5000, 5000) = P(s = 0) = \sqrt{\frac{2}{\pi N}} e^0 = \sqrt{\frac{2}{\pi 10000}} = 8 \times 10^{-3} \)

4b) \( P(6000, 4000) = P(s = 1000) = \sqrt{\frac{2}{\pi 10000}} e^{-2 \frac{1000^2}{10000}} = P(s = 0) \times e^{-200} \)

4c) We did the hard work for this problem in question 3c. Set \( N = 10000 \) there to get \( \delta N_{rms} = 50 \) and \( \frac{\delta N_{rms}}{N} = 5 \times 10^{-3} \).