Problem Set ---Due March 3

1) A Stern-Gerlach apparatus oriented along the z axis picks out a spin up state of an electron. The electron passes through a second Stern-Gerlach apparatus this one oriented along an axis which makes an angle $\theta$ relative to the z axis in y-z plane. What is the probability that this second Stern-Gerlach apparatus will find the electron in a spin up state?

2) The purpose of this problem is convince you that rotations in quantum mechanics about an axis $\hat{n}$ and through an angle $\theta$ are described by a rotation operator

$$\hat{R}(\theta) = e^{i\hat{J} \theta/\hbar}$$

where $\hat{J}$ is the quantum mechanical operator for the angular momentum (there is a notation infelicity here in that the hat has two meanings). If this is correct then a rotation of the about the z axis will rotate the $\hat{x}$ into a linear combination of $\hat{x}$ and $\hat{y}$:

$$\hat{R}_z(\theta) \hat{x} = \hat{x} \cos(\theta) + \hat{y} \sin(\theta) .$$

Verify that this relation is true. Hints: i) acting on $\hat{x}$ or $\hat{y}$ the angular momentum or relevance is the $x$ and $y$; ii) the relation holds if it holds at $\theta = 0$ and if the derivative of the right hand side matches that of the left hand side for all angles.

3) Consider rotations acting on spin $\frac{1}{2}$ states. In that case $\hat{R}(\theta) = e^{i\theta \hat{\sigma}_z /\hbar} = e^{i\theta \hat{\sigma}_z /2}$.

a) As a first step show that $\left| \hat{\sigma} \cdot \hat{n} \right|^2 = \mathbb{I}$.

b) Show that $e^{i\theta \hat{\sigma}_z /2} = \cos(\theta /2) + i(\hat{\sigma} \cdot \hat{n}) \sin(\theta /2)$. (Hint: expand both sides as a Taylor series in the angle and use a)

c) If this correctly gives the rotations then a rotation around the z axis through $\theta$ acting on $\hat{\sigma}_x$ should give a linear combination of $\hat{\sigma}_x$ and $\hat{\sigma}_y$:

$$\hat{R}_z(\theta) \hat{\sigma}_x R(\theta) \equiv e^{-i\theta \hat{\sigma}_x /2} \hat{\sigma}_x e^{i\theta \hat{\sigma}_x /2} = \hat{\sigma}_x \cos(\theta) + \hat{\sigma}_y \sin(\theta)$$

Hint: use part b) to do this.

4) In magnetic resonance experiments, the set up involves a strong constant magnetic field in the z direction and a rotating magnetic field in the x-y plane:

$$\vec{B} = B_0 \hat{z} + B'(\hat{x} \cos(\theta t) - \hat{y} \sin(\theta t)) .$$

a) Show that the time-dependent Schrödinger equation for the spinor of spin $\frac{1}{2}$ particle in such a field is given by

$$\left\{ - \gamma B_0 \hat{\sigma}_z - \gamma B'(\hat{\sigma}_x \cos(\theta t) - \hat{\sigma}_y \sin(\theta t)) \right\} \chi = 2i \frac{d\chi}{dt} .$$

b) It is common to work “in a rotating frame”: let $\chi' = e^{i\theta \hat{\sigma}_z /2} \chi$. Use the results of problem 3 c) to show that the time-dependent Schrödinger equation for the
The spinor in the rotating frame is given by 
\[
-\gamma B_{\text{eff}} \sigma_z - \gamma B' \sigma_x \right) |\psi\rangle' = 2i \frac{d|\psi\rangle'}{dt}
\]
with
\[
B_{\text{eff}} = B_0 - \frac{\omega}{\gamma}.
\]
Note that in this rotating frame the magnetic field looks constant. When 
\(B_{\text{eff}} \gg B'\) the effective magnetic field is essentially along the z axis and a spin up state will to good approximation remain spin up. However when \(B_{\text{eff}}\) goes to zero the magnetic field in the rotating frame is in the x direction and cause the spin to precess about the x axis---effectively flipping the spin from up to down continuously and causing the time average of the expectation value of \(\hat{s}_z\) to vanish.

c) Use the results of the precession calculation that we did in class (or you found in the book) to show that if the particle starts at \(t=0\) in a spin up state, the time averaged expectation value of \(\hat{s}_z\) is given by
\[
\langle \hat{s}_z \rangle = \frac{\hbar}{2} \frac{(B_0 - \frac{\omega}{\gamma})^2}{B_0^2 + B^2}.
\]
This quantifies the previous discussion. Away from the resonance the system is polarized in the z direction. Exactly on resonance the polarization in the z direction averages to zero.