plug in:

\[ E_{nls}^{(1l)} = \frac{E_n}{2m_c^2} \left[ 3 - \frac{4n}{j+\frac{1}{2}} \right] \]

\[ l = j - \frac{1}{2} \]

\[ l = j + \frac{1}{2} \]

Some result!!

\[ E_{ls} = \frac{E_n}{2m_c^2} \left[ 3 - \frac{4n}{2j+\frac{3}{2}} \right] \]

\( r_s \) depends on \( j \) but not \( l \)!

two states with same \( j \) but different \( l \) degenerate at this order

- split by QED Lamb shift

--- Idea of Zeeman effect:

Put atom in external mag field

Do pert. theory.
Time-dependent Perturbation Theory

Schrödinger Equation,

\[ H \psi = i \hbar \frac{\partial}{\partial t} \psi \]

explicitly \[ H(\vec{r}, t) \psi(\vec{r}, t) = i \hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) \]

If the potential is independent of time,

\[ V(\vec{r}, t) = V(\vec{r}) \]

The time-dependent part of the wave function can be separated out

\[ \psi(\vec{r}, t) = \psi(\vec{r}) e^{-i \frac{E}{\hbar} t} \]

\( \Rightarrow \) time-independent Schrödinger eq.

\[ H(\vec{r}) \psi(\vec{r}) = E \psi(\vec{r}) \]

Remarks:
1. Time-dependence is trivial
2. Not contribute to the probability \( \langle \psi | \psi \rangle \)
3. Linear combinations of these stationary states may have some time-dependence, but not Energy.
Time-dependent potential can lead to transitions between different states (energy).

Classical dynamics

Of course, if we can solve the time-dependent Schrödinger equation, we get the answers, and know how the transition between states happens...

However, in most cases, the time-dependent Schrödinger equation is very difficult to solve. We need to use perturbation method to get the approximate results. Assuming the time-dependent part of the Hamiltonian is small, just like the time-independent perturbation theory we have done before:
Example: Two-Level System

Suppose

\[ H(t) = H_0 + H'(t) \]

And two states

\[ H_0 |2a\rangle = E_a |2a\rangle \quad H_0 |2b\rangle = E_b |2b\rangle \]

\[ = E_a |a\rangle \quad = E_b |b\rangle \]

With

\[ \langle a | b \rangle = \delta_{ab} \quad \text{(orthonormal)} \]

Any state can be expressed as a linear combination of these two states,

\[ | \Psi \rangle = C_a |a\rangle + C_b |b\rangle \]

Where \[ |C_a| \] represents the probability that particle is in state \[ |a\rangle \],

\[ |C_a|^2 + |C_b|^2 = 1 \quad \text{Normalization} \]

Extending to time-dependent

\[ | \Psi(t) \rangle = C_a(t) |a\rangle + C_b(t) |b\rangle \]
The time-dependence of the wave function is represented by the time-dependence of the coefficient functions \( C_a(t) \) and \( C_b(t) \).

More explicitly, we write

\[
\Psi(t) = C_a(t) e^{-iE_a t/\hbar} |a\rangle + C_b(t) e^{-iE_b t/\hbar} |b\rangle
\]

Now, the goal is to solve the functions \( C_a(t) \), \( C_b(t) \).

Schrödinger equation

\[
\hat{H} \Psi(t) = i \hbar \frac{\partial}{\partial t} \Psi(t) \quad H = H_0 + H'(t)
\]

Let

\[
\hat{H} \Psi(t) = \left( \hat{H}_0 + \hat{H}'(t) \right) \left[ C_a(t) e^{-iE_a t/\hbar} |a\rangle + C_b(t) e^{-iE_b t/\hbar} |b\rangle \right]
\]

Equation 3:

\[
\frac{\partial}{\partial t} \Psi(t) = \left( \hat{H}_0 + \hat{H}'(t) \right) \left[ C_a(t) e^{-iE_a t/\hbar} |a\rangle + C_b(t) e^{-iE_b t/\hbar} |b\rangle \right]
\]

\[
\frac{\partial}{\partial t} \Psi(t) = \frac{i\hbar}{2} \left[ C_a(t) e^{-iE_a t/\hbar} |a\rangle + C_b(t) e^{-iE_b t/\hbar} |b\rangle \right]
\]

\[
\frac{\partial}{\partial t} \Psi(t) = \frac{i\hbar}{2} \left[ \left( -iE_a/\hbar \right) C_a(t) e^{-iE_a t/\hbar} |a\rangle + \left( -iE_b/\hbar \right) C_b(t) e^{-iE_b t/\hbar} |b\rangle \right]
\]

\[
\Psi(t) = \left[ -iE_a/\hbar C_a(t) e^{-iE_a t/\hbar} |a\rangle + \left( -iE_b/\hbar \right) C_b(t) e^{-iE_b t/\hbar} |b\rangle \right]
\]

\[
C_a(t) e^{-iE_a t/\hbar} |a\rangle + C_b(t) e^{-iE_b t/\hbar} |b\rangle
\]
\[ \dot{C}_a(t)e^{-\frac{iE_a t}{\hbar}} |a\rangle + \dot{C}_b(t)e^{-\frac{iE_b t}{\hbar}} |b\rangle \]

\[ \frac{-i}{\hbar} \left[ \frac{C_a(t)}{E_a} e^{-\frac{iE_a t}{\hbar}} H'(+) |a\rangle + C_b(t) e^{-\frac{iE_b t}{\hbar}} H'(+) |b\rangle \right] \]

Left product with \( |a\rangle \)

\[ \dot{C}_a(t) = \frac{i}{\hbar} \left[ \frac{C_a(t)}{E_a} \langle a | H'(+)|a\rangle + C_b(t) \langle a | H'(+)|b\rangle \right] \]

\[ = \frac{i}{\hbar} \left[ H_{a\alpha}(+) C_a(t) + H_{a\beta}(+) e^{-\frac{iE_a t}{\hbar}} C_b(t) \right] \]

Similarly

\[ \dot{C}_b(t) = \frac{i}{\hbar} \left[ H_{b\alpha}(+) C_b(t) + H_{b\beta}(+) e^{-\frac{iE_b t}{\hbar}} C_a(t) \right] \]

Comments on the Matrix \( H_{a\beta}(+) \)

1. It's time-dependent.

2. Hermiticity, \( H_{a\beta} = (H_{\beta a})^* \)

3. \( H \) mixes states.

4. \( |C_a(t)|^2 + |C_b(t)|^2 = 1 \) for any \( t \).
Example:

An electron in a time-dependent Magnetic Field:

\[ H(t) = - \gamma B(t) \cdot \vec{S} \]

\( \vec{S} \) is the spin for the electron,

\[
\begin{align*}
S_z &= \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
S_x &= \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
S_y &= \frac{\hbar}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}
\end{align*}
\]

Suppose the dominant Magnetic Field is in z direction,

\[ H_o = -\gamma B_0 S_z \] is time-independent.

Eigenstates of \( H_0 \):

\[ 1^+ >, \quad 1^- > \]

\[ H_0 |1^+ > = -\frac{\gamma B_0 \hbar}{2} |1^+ > \]

\[ H_0 |1^- > = \frac{\gamma B_0 \hbar}{2} |1^- > \]

\[ E_{1^+} = -\frac{\gamma B_0 \hbar}{2} \quad E_{1^-} = \frac{\gamma B_0 \hbar}{2} \]

\[ \omega_o = (E_{1^-} - E_{1^+})/\hbar = \gamma B_0 \]
Now, we add some perturbation, varying the magnetic field. 

1) \( B(t) \) in z direction

2) \( B(t) \) in x direction

\( \hat{H}'(t) = -\gamma B_z(t) \hat{S}_z \) — self- mixing

\[ H'_{a0} = 0 = \langle a | \hat{H}'(t) | b \rangle = \langle a | \hat{H}'(t) | 1 \rangle = \langle a | \hat{S}_z | 1 \rangle = 0 \]

\[ H'_{aa} = \langle a | \hat{H}'(t) | a \rangle = \langle a | \hat{H}'(t) | 1 \rangle = -\frac{\gamma}{2} B_z(t) \]

\[ H'_{bb} = \frac{\gamma}{2} B_z(t) \]

\[ C_a(t) = \frac{i}{\hbar} H'_{aa} C_a(t) \]

\[ = -\frac{i}{\hbar} \gamma B_z(t) C_a(t) \]

\[ \Psi(t) = C_a(0) e^{\frac{i}{\hbar} \gamma B_z(t) \frac{t}{\hbar} \frac{1}{2} - i E_a t/\hbar} [1]\]

\[ + C_b(0) e^{-\frac{i}{\hbar} \gamma (B_z(t) + B_0) \frac{t}{\hbar} \frac{1}{2} - i E_b t/\hbar} [1]\]

\[ = C_a(0) e^{\frac{i}{\hbar} \gamma (B_z(t) + B_0) \frac{t}{\hbar} \frac{1}{2} - i E_a t/\hbar} + C_b(0) e^{-\frac{i}{\hbar} \gamma (B_0 + B_z(t)) \frac{t}{\hbar} \frac{1}{2} - i E_b t/\hbar} \]
i) Check \[ |Ca(+)|^2 + |Cb(+)|^2 = 1 \]

ii) if \[ Ca(0) = 1 \quad Cb(0) = 0 \]

\[ |Ca(\pm)| = 1 \quad Cb(\pm) = 0 \]

No transition.

iii) \[ \langle S_2 \rangle = \langle 24(+) | S_2 | 24(+) \rangle \]

\[ = |Ca(+)|^2 \langle a1 S_2 | a \rangle - |Cb(+)|^2 \langle b1 S_2 | b \rangle \]

\[ = \left( |Ca(0)|^2 - |Cb(0)|^2 \right) \frac{1}{2} \]

is independent of time.

iv) Energy \[ \langle \mathcal{H} \rangle = \langle 24(+) | \mathcal{H} | 24(+) \rangle \]

\[ = |Ca(+)|^2 \langle a1 \mathcal{H} | a \rangle + |Cb(+)|^2 \langle b1 \mathcal{H} | b \rangle \]

\[ = Ca(0)^2 \left( -\frac{\mathcal{H}}{2} (B_0 + B_2(\pm)) \right) + Cb(0)^2 \left( \frac{\mathcal{H}}{2} (B_0 + B_2(\pm)) \right) \]

\[ = \left( Cb(0)^2 - Ca(0)^2 \right) \left( \frac{\mathcal{H}}{2} B_0 \pm + \frac{\mathcal{H}}{2} B_2(\pm) \right) \]

depends on time, but the dependence is trivial.
2) \( H'(t) = -\gamma B x(t) \hat{S}_x \)

\[
H_{aa}' = -\gamma B x(t) \langle a_1 \hat{S}_x a \rangle = -\gamma B x(t) \langle t \hat{S}_x t \rangle = 0
\]

\( H_{bb}' = 0 \)

\( H_{ab}' = -\gamma B x(t) \langle t \hat{S}_x t \rangle = -\frac{\gamma}{2} B x(t) = H_{aa}(t) \)

\[
\dot{C}_a(t) = \frac{-i}{\hbar} H_{ab}(t) e^{-i(E_a-E_b)t/\hbar} C_b(t) = \frac{i}{2} \gamma B x(t) e^{-i(E_a-E_b)t/\hbar} C_b(t)
\]

\[
\begin{align*}
\dot{C}_a(t) &= \frac{i}{2} \gamma B x(t) e^{-i \lambda(t)} C_b(t) \\
\dot{C}_b(t) &= \frac{i}{2} \gamma B x(t) e^{i \lambda(t)} C_a(t)
\end{align*}
\]

This is a more general case, and it is difficult to solve exactly.

We need to do perturbation.
Time-dependent perturbation theory.

We have the case, \( \hat{H}_0 + \delta \equiv \hat{H}_0 + \hat{H}_d \neq 0 \) but \( \hat{H}_d \neq 0 \).

\[
\begin{align*}
\dot{C}_a(t) &= -\frac{i}{\hbar} \hat{H}_{d0}(t) e^{-i(E_0 - E_a)/\hbar} C_b(t), \\
\dot{C}_b(t) &= -\frac{i}{\hbar} \hat{H}_{d0}(t) e^{-i(E_0 - E_b)/\hbar} C_a(t).
\end{align*}
\]

Define \( \omega_0 = E_b - E_a \).

Suppose at \( t = 0 \), \( C_a(0) = 1 \), \( C_b(0) = 0 \).

0) Zeroth order

\( \hat{H}' = 0 \), \( \hat{H}_{d0} = 0 \),

\( C_a(t) = 1 \), \( C_b(t) = 0 \) for any \( t \).

1) First order,

Substitute zeroth order result to the right-hand sides of the equations,

\[
\begin{align*}
\dot{C}_a(t) &= 0 \quad \Rightarrow \quad C_a(t) = 1 \\
\dot{C}_b(t) &= -\frac{i}{\hbar} \hat{H}_{d0}(t) e^{i \omega_0 t} \\
C_b(t) &= -\frac{i}{\hbar} \int_0^t \hat{H}_{d0}(t') e^{i \omega_0 t'} dt'.
\end{align*}
\]
2) Second order:
Substitute the first order results to the right hand sides of the equations

\[
\begin{align*}
\dot{C}_a(t) &= -\frac{i}{\hbar^2} H'_{ab}(t) e^{-i\omega_1 t} \left[ -\frac{i}{\hbar} \int_0^t H'_{ba}(\tau) e^{i\omega_0 \tau} d\tau \right] \\
\dot{C}_b(t) &= -\frac{i}{\hbar} H_{ba}(t) e^{i\omega_0 t} \\
\end{align*}
\]

\Rightarrow \quad C_a(t) = 1 - \frac{1}{\hbar^2} \int_0^t H_{ab}(t') e^{-i\omega_0 t'} \left[ \int_0^t H'_{ba}(t'') e^{i\omega_0 t''} dt'' \right] dt'

\Rightarrow \quad C_b(t) = C_b(t) \text{ at first order}

W) Nth order:
Substitute the (N-1)th order results to the right hand sides of the equations.

\Rightarrow \text{ get the Nth order results are integrals of (N-1)th results}

Remarks: (1) order by order

(2) Ca is modified by even order
C6 is modified by odd order
(3) Although for every order

\[ |C_a(t)|^2 + |C_b(t)|^2 = 1 \]

For every power of \( H' \), \( |Ca(t)|^2 + |Cb(t)|^2 = 1 \)

e.g. Zero power of \( H' \)

\[ C_a(t) = 1 \quad C_b(t) = 0 \quad \text{correct} \]

One power of \( H' \)

None

Two power of \( H' \)

\[ |Ca(t)|^2 + |Cb(t)|^2 \]

\[ = 1 - \frac{2}{\hbar^2} \int_0^t H_a(t') e^{-i\omega_0 t'} \left[ \int_0^{t'} H_b(t'' e^{i\omega_0 t''}) dt'' \right] dt' \]
\[ + \frac{1}{\hbar^2} \left( \int_0^t H_b(t') e^{-i\omega_0 t'} dt' \right)^2 \]
\[ + \frac{1}{\hbar^2} \left( \int_0^t H_b(t') e^{-i\omega_0 t'} dt' \right)^2 \]
\[ \int_0^t H_a(t') e^{i\omega_0 t'} dt' \int_0^t H_a(t') e^{i\omega_0 t'} dt' \]

\[ = 1 \quad \text{correct} \]
Back to our example:

\[
\begin{align*}
\dot{C}_a(t) &= \frac{i}{2} \gamma B_x(t) e^{-i \gamma B_o t} C_a(t) \\
\dot{C}_b(t) &= \frac{i}{2} \gamma B_x(t) e^{i \gamma B_o t} C_a(t)
\end{align*}
\]

Initial state is spin up \( \langle + \rangle \), like ground state

\[
C_a(0) = 1, \quad C_b(0) = 0
\]

Zeroth order: \( C_a(t) = 1, \quad C_b(t) = 0 \)

First order: \( C_a(t) = 1 \)

\[
C_b(t) = \frac{i}{2} \gamma \int_0^t B_x(t') e^{i \gamma B_o t'} dt' \neq 0
\]

(transition from spin up state to spin down state)

Second order:

\[
C_a(t) = 1 - \frac{\gamma^2}{4} \int_0^t B_x(t') e^{-i \gamma B_o t'} dt' \left[ \int_0^t B_x(t'') e^{i \gamma B_o t''} dt'' \right] dt'
\]

\[
C_b(t) = \frac{i}{2} \gamma \int_0^t B_x(t') e^{i \gamma B_o t'} dt'
\]

Simple case, if \( B_x(t) = \alpha \delta(t-t_0) \)

To the first order \( C_a(t) = 1 \)

\[
C_b(t) = \begin{cases} 0 & t < t_0 \\
\frac{i}{2} \gamma e^{i \gamma B_o t} & t = t_0 \\
\frac{i}{2} \gamma e^{-i \gamma B_o t} & t > t_0
\end{cases}
\]

\[
\text{Step function}
\]
Now, if we measure the spin of the electron along the \( \hat{z} \) direction, it has \( \frac{1}{4} \alpha^2 \) chance with \( S_z = -\frac{1}{2} \) to.

For \( t > t_0 \)

\[
\langle S_z \rangle = (|K_a(\tau)|^2 - |K_b(\tau)|^2) \frac{T_0}{2}
\]

\[
= (1 - \frac{1}{4} \alpha^2 \delta^2) \frac{T_0}{2}
\]

\[
= \left(1 - \frac{1}{2} \alpha^2 \delta^2\right) \frac{T_0}{2} + o(\alpha^2)
\]

and the energy

\[
\langle E \rangle = \langle H \rangle = \langle H_0 + H' \rangle
\]

\[
= \langle -\frac{1}{2} B_0 S_z^2 - \alpha \delta (\hat{z} \cdot \hat{r}) S_z \rangle
\]

\[
= -\delta B_0 \langle S_z \rangle = -\frac{\delta B_0 T_0}{2} \left(1 - \frac{1}{2} \alpha^2 \delta^2\right)
\]

\[+ o(\alpha^2)\]
**Sinusoidal Perturbation**

This is the most interesting case.

Suppose the perturbation has sinusoidal time dependence

\[ H'(r, t) = V(r) \cos(\omega t) \]

\[ \Rightarrow \quad H_{ab}(t) = V_{ab} \cos(\omega t) \]

\[ V_{ab} = \langle a | V | b \rangle \]

To the first order accuracy,

\[ C_{a}(t) = 1 \]

\[ C_{b}(t) = -\frac{i}{\hbar} \int_{0}^{t} H_{ab}(t') e^{i\omega t'} dt' \]

\[ = -\frac{i}{\hbar} V_{ab} \int_{0}^{t} \cos(\omega t) e^{i\omega t'} dt' \]

\[ = -\frac{i}{2\hbar} V_{ab} \left[ \int_{0}^{t} e^{i(\omega t + \omega t')} + e^{i(\omega t - \omega t')} \right] dt' \]

\[ = \frac{V_{ab}}{2\hbar} \left[ \frac{1 - e^{i(\omega t)} + 1 - e^{-i(\omega t)}}{\omega} \right] \]

If \( \omega \) is close to \( \omega_0 \), drop the first term

\[ = \frac{V_{ab}}{2\hbar} \frac{e^{-i(\omega - \omega_0)t/2} - e^{-i(\omega + \omega_0)t/2}}{-i(\omega_0)\hbar} e^{i(\omega_0 - \omega)t/2} \]
\[ C_0(t) = -\frac{\text{V}_{ab}}{\hbar^2} \frac{i^2 e^{(\omega_0 - \omega)t/2} e^{i(\omega - \omega_0)t/2}}{\omega_0 - \omega} \]

which is a sinusoidal function.

The probability of the transition from \( |a\rangle \) to \( |b\rangle \):

\[ P_{a \rightarrow b}(t) = \left( \frac{\text{V}_{ab}}{\hbar} \right)^2 \frac{e^{i(\omega_0 - \omega)t/2}}{(\omega_0 - \omega)^2} \]

It oscillates sinusoidally as a function of time.

![Graph showing oscillatory behavior over time with much less than 1 perturbation.]

Remarks:

1. \( P_{a \rightarrow b} \) must be much less than 1
2. \( t_n = \frac{2\pi}{|\omega_0 - \omega|} \), \( P_{a \rightarrow b} = 0 \)
3. \( P_{a \rightarrow b} \) as a function of \( \omega \)

Peaks at \( \omega = \omega_0 \) with \( \left( \frac{\text{V}_{ab}}{2\hbar} \right)^2 \) height, width \( 4\pi/\epsilon \).
Since perturbation works only for \( P_{\text{max}} < 1 \), it must be small.

Back to our example, electron in a time-dependent magnetic field

\[ H = -\gamma B_z \hat{S}_z - \gamma B_x(t) \hat{S}_x \]

Assume \( B_x(t) = \alpha \cos(wt) \), \( \alpha \) is a small parameter

at \( t = 0 \), \( \psi(0) = 1 \), \( \psi_b(0) = 0 \).

\[ \psi(t=0) = |\psi_0\rangle = |t\rangle \]

First order perturbation

\[ \psi_a(t) = 1 \]

\[ \psi_b(t) = \frac{-i}{\hbar} \int_0^t H_a(t') \psi_a(t') dt' \]
\[ H'(t) = -\gamma P_x(t) \hat{S}_x = -\gamma \hat{S}_x \propto \cos \omega t. \]

\[ H_{2s} = 0 = H_{s0} \]

\[ H_{c0} = -\frac{i}{2} \gamma \alpha \cos \omega t = V_{ab} \cos \omega t \]

\[ \Rightarrow C_b(t) = -i \frac{V_{ab}}{\hbar} \frac{\gamma \alpha}{\omega - \omega_c} \left( \frac{e^{i(\omega_0 - \omega)t/2}}{\omega_0 - \omega} \right) = i \frac{\gamma \alpha}{\hbar} \frac{\gamma \alpha}{\omega_0 - \omega} \left( \frac{e^{i(\omega_0 - \omega)t/2}}{\omega_0 - \omega} \right) \]

\[ \omega_0 = \gamma B_0 \]

\[ P_{a \rightarrow b}(t) = |C_b(t)|^2 = \frac{(\gamma \alpha)^2}{2(\omega_0 - \omega)} \]

\[ \text{Underlying Physics: Absorbing Radiation.} \]
Limitation of treatment in book

- two laws
- \( C \{ \psi \} = \psi \{ H \} \psi = 0 \)

O.K. for probabilities at 1st order
more general treatment better done formally

\[
\hat{H}(t) \ 1 \psi(t) = i \hbar \frac{\partial}{\partial t} \ 1 \psi(t)
\]

now in general

\[
1 \psi(t) = \hat{U}(t) \ 1 \psi(0)
\]

\[\downarrow\]

unitary (time evolution op)
why?

so

\[
\hat{H} \ 1 \psi(t) = i \hbar \frac{\partial}{\partial t} \ 1 \psi(t)
\]

true for all \( 1 \psi(0) \) so as

operator \( \hat{H} \ \hat{U}(t) = i \hbar \frac{\partial}{\partial t} \ \hat{U}(t) \)
cute factoid — all of QM is expressed in
terms of matrix elements all else is unphysical
(e.g. wavefunction)

consider \[ |\psi(t)\rangle = \hat{U}(t) |\phi(\infty)\rangle \]
\[ |\phi(t)\rangle = \hat{U}(t) |\phi(\infty)\rangle \]

what is

\[ \langle \psi(t) | \hat{A} | \phi(t) \rangle \]

which now I can think of this in

two ways

Schrödinger picture is as written
\[ \hat{A} \] is time-independent
states are time-dependent

but

\[ \langle \psi(t) | \hat{A} | \phi(t) \rangle = \langle \psi(0) | \hat{U}(t)^\dagger \hat{A} \hat{U}(t) | \phi(0) \rangle \]

define \[ \hat{A}_H \] (Heisenberg rep) as

\[ \hat{A}_H \psi = \hat{U}^\dagger(t) \hat{A} \hat{U}(t) \psi \]

now we have
\[ \langle \psi(t) | \hat{A} | \phi(t) \rangle = \langle \psi(0) | \hat{A}_H(t) | \phi(0) \rangle \]

Heisenberg picture
- wave functions time independent
- all time evolution is in operators

Simple case: \( \hat{A} \) is time independent

\[ \hat{H} \hat{U}(t) = i \hbar \frac{\partial}{\partial t} \hat{U}(t) \]

Formal solution \( \hat{U} = e^{-i \hat{H} t / \hbar} \), what does this mean?

Proof: plug & chug.

Now this is very simple in eigenbasis of \( \hat{A} \)

\[ \hat{H} | \psi_i \rangle = E_i | \psi_i \rangle \]

\[ \langle \psi_i | \hat{A} | \psi_i \rangle = \delta_{ij} e^{-i \epsilon_j t / \hbar} \]

\[
\begin{pmatrix}
e^{-i \epsilon_1 t / \hbar} \\
e^{-i \epsilon_2 t / \hbar}
\end{pmatrix}
\]
Note this does not work if $\hat{H}$ is time-independent.

Why?

Heisenberg picture is like using a time-dependent basis.

\[
\langle \psi(0) | A^\dagger | \phi(0) \rangle = \langle \psi(0) | u^* A^\dagger u \ u^T | \phi(0) \rangle 
\]

Note $u^T | \phi(t) \rangle = \mathbf{a} | \phi(t) \rangle$.

\[
\langle \psi(t) | A | \phi(t) \rangle = \langle \psi(t) | u \ u^T A^\dagger u_0 \ u^* \phi(t) \rangle 
\]

Now suppose $\hat{H} = \hat{H}_0 + \hat{H}'$ with $\hat{H}_0$ time-independent.

Let us define $u_0 = e^{-i \hat{H}_0 t_0}$.

Any matrix element may be written as

\[
\langle \psi(t) | A | \phi(t) \rangle = \langle \psi(t) | u \ u^T A^\dagger u_0 \ u^* \phi(t) \rangle 
\]

$\Rightarrow \dot{A}_x | \phi(t) \rangle \Rightarrow$ interaction picture suitable for td part theory

with $\dot{A}_x = u^* \dot{u} A u_0 | \phi(t) \rangle$.

\[
\langle \psi(t) | \phi(t) \rangle = \langle \psi(t) | u^* | \phi(t) \rangle 
\]

Removes the time-dependence due to $\hat{H}_0$ for $| \psi \rangle$ but leaves time-dependence due to $\hat{H}'$. 
Now let us find time evolution of $|\psi\rangle$, i.e., $|\psi\rangle$ in interaction rep

\[ \frac{i\hbar}{\sqrt{2}} |\psi\rangle = \frac{i\hbar}{\sqrt{2}} u_0^+(t) |\psi\rangle \]

\[ = \left( \frac{du_0}{dt} \right) |\psi\rangle + i\hbar u_0^+ \frac{\partial |\psi\rangle}{\partial t} \]

but $i\hbar \frac{du_0}{dt} = H_0 u_0$.

\[ \implies \]

\[ -i\hbar \frac{du_0^+}{dt} = u_0^+ H_0 \]

\[ \frac{\partial |\psi\rangle}{\partial t} = \frac{\partial |\psi\rangle}{\partial t} - H = H_0 + H_0' \]

\[ = \frac{\partial}{\partial t} \left( |\psi\rangle \right) - H = H_0 + H_0' \]

\[ \implies \]

\[ \frac{\hbar^2}{2m} |\psi\rangle = \left[ -u_0^+ H_0 |\psi\rangle + u_0^+ \left( H_0 + H_0' \right) |\psi\rangle \right] \]

\[ = u_0^+ H_0' |\psi\rangle \]

\[ = u_0^+ H_0' u_0 |\psi\rangle \]

\[ = H_{\text{int}} |\psi\rangle \]
so form of Schrödinger in interaction picture is same as usual with $H_I$ playing role.

Hamiltonian

so

$|\psi(t)\rangle = u_I(t) |\psi(t)\rangle$

Now solving for $u_I(t)$ gives the full time evolution since

$u^+\psi(t)\rangle = |\psi(t)\rangle$

or

$|\psi(t)\rangle = u_0(t) |\psi(t)\rangle$

$= u_0(t) u_I(t) |\psi(t)\rangle$

Now let us solve for $u_I(t)$ perturbatively

so

$c t \frac{\partial}{\partial t} u_I(t) |\psi(t)\rangle = H_I u_I(t) |\psi(t)\rangle$

true for all $|\psi(t)\rangle$ so

$c t \frac{\partial}{\partial t} u_I = H_I u_I$
Now we can solve for $u_\perp$ perturbatively.

Say that $H_x \rightarrow \lambda H_x$ and count powers of $\lambda$:

$$u_\perp = 1 + \lambda u_\perp^{(1)} + \lambda^2 u_\perp^{(2)} + \lambda^3 u_\perp^{(3)} + \cdots$$

If $H_x = 0$, $u_\perp = 1$

$$\lambda^{\lambda x} (1 + \lambda u_\perp^{(1)} + \lambda^2 u_\perp^{(2)} + \cdots) = e^{\lambda x} \frac{2}{\lambda x} (\lambda^{\lambda x} + \lambda^{\lambda x} u_\perp^{(1)} + \cdots)$$

$$\lambda^{\lambda x} \frac{\partial}{\partial t} (\lambda^{\lambda x} + \lambda^{\lambda x} u_\perp^{(1)} + \cdots) = e^{\lambda x} \frac{2}{\lambda x} \lambda^{\lambda x} \frac{\partial}{\partial t} \left( \lambda^{\lambda x} + \lambda^{\lambda x} u_\perp^{(1)} + \cdots \right)$$

So

$$u_\perp^{(1)} = \frac{1}{\lambda} \int_0^t dt' H_x (t')$$

integrate 1st term

$$u_\perp^{(2)} = \frac{-i}{\lambda} \int_0^t dt' \int_0^{t'} dt'' H_x (t') u_\perp^{(1)} (t'') = \left( \frac{-i}{\lambda} \right)^2 \int_0^t dt' \int_0^{t'} dt'' H_x (t') \int_0^{t''} dt''' H_x (t'')$$

$$u_\perp^{(3)} = \frac{-i}{\lambda} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' H_x (t') u_\perp^{(1)} (t'') u_\perp^{(1)} (t''') = \left( \frac{-i}{\lambda} \right)^3 \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' H_x (t') \int_0^{t''} dt''' H_x (t') \int_0^{t''} dt''' H_x (t')$$

$$u_\perp^{(4)} = \frac{-i}{\lambda} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \int_0^{t'''} dt'''' H_x (t') u_\perp^{(1)} (t'') u_\perp^{(1)} (t'''') u_\perp^{(1)} (t''''') = \left( \frac{-i}{\lambda} \right)^4 \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \int_0^{t'''} dt'''' H_x (t') \int_0^{t'} dt''' H_x (t') \int_0^{t''} dt''' H_x (t') \int_0^{t'''} dt''' H_x (t')$$
Two level system

\[ \langle \hat{H}_1\hat{H}_1 \rangle = \langle \hat{H}_1\hat{H}_2 \rangle = 0 \]

\[ H = \nu \cos(\omega t) \]

\[ P_{a \rightarrow b} = \frac{|V_{ab}|^2}{\hbar^2} \frac{\sin^2 [(\omega_a - \omega) t / 2]}{(\omega - \omega_0)^2} \quad (1st \ order) \]

derived previously

answer is time-dependent

sensible to look at time average

\[ \overline{P_{a \rightarrow b}} = \frac{1}{2\pi} \frac{|V_{ab}|^2}{(\omega - \omega_0)^2} \]

gives a way to measure \( \nu \) at \( \nu_0 \)

- big for \( \omega \neq \omega_0 \) why?

relation to photon

\[ \nu \]
what if \( w = w_0 \) blows up!

Clearly this is wrong

why - 1st order part breaks down then even for small \( w \)

when is 1st order O(1) 

for \( \frac{P_{\omega_0}}{P_{\omega}} \ll 1 \)

when \( w = w_0 \) does 1st order part always fail? 

Yes for \( \frac{P_{\omega_0}}{P_{\omega}} \)

but \( P_{\omega_0}(t) \) is O(1) for small time

\[
P_{\omega_0}(t) = \frac{1}{4} \frac{1}{t^2} \left( \frac{\sin^2 \left( \frac{(w - w_0)^2 t}{2} \right)}{(w - w_0)^2} \right)
\]

\[
= \frac{1}{4} \frac{1}{t^2} \frac{(w - w_0)^2 + \frac{t^2}{4}}{(w - w_0)^2}
\]

\[
= \frac{1}{4} \frac{1}{t^2} \frac{w^2 + \frac{t^2}{4}}{(w - w_0)^2}
\]

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this is ok for \( P_{a\rightarrow b}(t) \ll 1 \)

i.e. \( t \ll \frac{v_{ab}^2}{10|v_{ab}|^2} \)

actually even when \( u \) different from \( u_{ab}(t) \) eventually becomes bad but \( \Gamma_{ab} \) is still ok, why.

Key issue is time scale

Now this suggests another type of approximation method instead of looking at size of perturbation look at its speed — how fast \( H' \) changes.

**Mysterious**

Fast or slow compared to what?

\( \Gamma' \) is time scale over which \( H' \) changes. \( \Gamma \) is some \( \exp(-t/\Gamma) \) to –

issue is \( \Theta(E_a - E_b)/\Gamma' \gg \Gamma^{-1} \)

or \( \Theta(E_a - E_b)/\Gamma' \ll \Gamma^{-1} \)

for all a, b \( \mu \tau \).
First case: \((E_n - E_0)/\hbar >> \tau^{-1}\) is the adiabatic region.

\((E_n - E_0)/\hbar << \tau^{-1}\) is the intermediate sudden regime.

Typical problem: \(\hat{H}(t)\) starts at \(\hat{H}_i\) and ends at \(\hat{H}_f\);

\[\hat{H}(t = -\infty) = \hat{H}_i,\]
\[\hat{H}(t = +\infty) = \hat{H}_f,\]

with \(\hat{H}_i \neq \hat{H}_f\).

Now in this case it is not always clear what basis to work in: the eigenbasis of \(\hat{H}_i\) or \(\hat{H}_f\) or what.

E.g., we have an atom in its ground state and turn on a magnetic field.
Sudden approx is simple:

if $\hat{H}$ changes from $\hat{H}_i$ to $\hat{H}_f$ more rapidly, then all time scales it the problem ($\frac{\Delta t}{\Delta t} \gg \frac{1}{\tau}$), then wave function has no time to change during switch from $\hat{H}_i$ to $\hat{H}_f$

call the time of switch $t = t_0$

Equation of motion

$\hat{H}_c |\psi\rangle = i\hbar \frac{\partial}{\partial t} |\psi\rangle$ for $t < t_0$

$\hat{H}_c |\psi\rangle = i\hbar \frac{\partial}{\partial t} |\psi\rangle$ for $t > t_0$

Solution:

$|\psi(t)\rangle = e^{-i\hat{H}_c t/\hbar} |\psi(0)\rangle$ for $t < t_0$

$|\psi(t)\rangle = e^{-i\hat{H}_c (t-t_0)} e^{-i\hat{H}_{t_0} t} |\psi(0)\rangle$ for $t > t_0$

Note $e^{-i\hat{H}_c (t-t_0)} e^{-i\hat{H}_{t_0} t} \neq e^{-i(\hat{H}_c (t-t_0) + \hat{H}_{t_0} t)}$ unless $[\hat{H}_c, \hat{H}_{t_0}] = 0$

In practice the way to do this is to work in $\hat{H}_i$ basis for $t < t_0$ and $\hat{H}_f$ basis for $t > t_0$.

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β decay problem on exam

Now this is intellectually easy to understand. What about the opposite limit, the slow

\[ \tau \gg \frac{t}{|E_0 - E_k|} \quad \text{for all } \mathbb{R} \cap \mathbb{R} \]

Basic idea is quite simple

adiabatic limit is opposite
Claim: in adiabatic limit state remains in an eigenstate of time local $H$ it began in and phase evolves according to

time dependent $H$

$|\psi(t)\rangle = e^{i\int_0^t H(t')dt'} |\psi(0)\rangle$

$\phi(t) = -\int_0^t \frac{E_i(t')}{\hbar} dt'$

intuitive argument

to make a transition need Fourier components $\sim \frac{\Delta E}{\hbar}$

but as $f(t) \Rightarrow \tilde{f}(\frac{t}{\hbar})$ stretches out as $t \rightarrow \infty$

$\tilde{f}(\omega) \Rightarrow \tilde{f}(\frac{1}{\hbar} \omega)$ Fourier spectrum gets squeezed as $t \rightarrow \infty$

no Fourier components of size comparable to $\Delta E$

system must stay in $\text{sinc}$ state with level very slowly evolving
Example

\[ H(t) = -\frac{i \omega}{2} \begin{pmatrix} 0 & e^{-i \omega t} \\ e^{i \omega t} & 0 \end{pmatrix} \quad \text{magnetic field rotating} \]

\[ B \text{ field with } \omega = \text{sgn } \left( \mathbf{B} \right) \]

and \[ \mathbf{B} = |\mathbf{B}| \left( x \cos(\omega t) + y \sin(\omega t) \right) \]

Exact Solution

\[ H \times = i \mathbf{H} \times \]

\[ \text{Claim } \mathbf{X}(t) = \begin{pmatrix} \frac{1}{2} e^{-i \omega t} \cos(\omega t) + \frac{i (\omega + \omega)}{2} \sin(\omega t) \\ \frac{1}{2} e^{i \omega t} \cos(\omega t) + \frac{i (\omega - \omega)}{2} \sin(\omega t) \end{pmatrix} \]

\[ d = \sqrt{\omega^2 + \omega^2} \]

check: plug x chang

at \[ \mathbf{X}(t) = \begin{pmatrix} \frac{1}{2} e^{-i \omega t} \\ \frac{1}{2} e^{i \omega t} \end{pmatrix} \]

\[ = \mathbf{X}(x) \]

\[ \text{Note:}\]

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jets look at

\[ H(0) = -tw, S_x \]

\[ H(t = \frac{\pi}{w}) = t, w, S_x \]

let us look at

\[ X(t = \frac{\pi}{w}) = \left( \frac{i}{\sqrt{2}} e^{-i \frac{\pi}{2}} \cos \left( \frac{1}{w} \frac{\pi}{2} \right) + \frac{i (\omega_1 + \omega)}{\lambda} \sin \left( \frac{1}{w} \frac{\pi}{2} \right) \right) \]

\[ \left( \frac{i}{\sqrt{2}} e^{i \frac{\pi}{2}} \left( \cos \left( \frac{1}{w} \frac{\pi}{2} \right) + \frac{i (\omega_1 - \omega)}{\lambda} \sin \left( \frac{1}{w} \frac{\pi}{2} \right) \right) \right) \]

first look in sudden limit,

\[ w \gg \omega, \quad (why \; is \; this \; sudden) \]

\[ \lambda = \sqrt{w^2 + \omega^2} \propto w \]

\[ \frac{\omega_1 - \omega}{\lambda} \approx \frac{1}{2} \]

\[ \frac{1}{\lambda} \approx \frac{1}{2} \]

so

\[ X(t = \frac{\pi}{w}) \approx \left( \frac{i}{\sqrt{2}} e^{-i \frac{\pi}{2}} \left( \cos \left( \frac{\pi}{2} \right) + i \sin \left( \frac{\pi}{2} \right) \right) \right) \]

\[ \left( \frac{i}{\sqrt{2}} e^{i \frac{\pi}{2}} e^{i \frac{\pi}{2}} \right) \]

\[ = \left( \frac{i}{\sqrt{2}} \right) = X(t = 0) \quad as \; it \; should. \]
Next, look in detail at the limit:

\[ w_i > > \omega \]

\[ l = \sqrt{w_1^2 + \omega^2} \propto \omega \]

\[ \frac{w_i}{\omega} \ll 1 \]

So

\[ X(t = \frac{t_0}{\omega}) \approx \left( \frac{i}{\hbar} e^{-i \frac{\pi}{2}} \left( \cos \left( \frac{\omega}{\hbar} t_0 \right) + i \sin \left( \frac{\omega}{2\hbar} t_0 \right) \right) \right) \]

\[ = \left( \frac{i}{\hbar} e^{i \frac{\pi}{2}} \right) \left( \frac{i}{\hbar} e^{i \frac{\pi}{2}} \right) \]

\[ = \left( \frac{i}{\hbar} \right) \left( \frac{i}{\hbar} \right) \left( \cos \left( \frac{\omega}{\hbar} t_0 \right) + i \sin \left( \frac{\omega}{2\hbar} t_0 \right) \right) = X \cdot \bar{X} \]

Phase probability that it is down in \( x \) direction is 1, but at \( t = 0 \), it was 0; with unit probability is sustained.

Now, \( H(x, -\pi) \) has its ground state as down.
"Proof" of adiabatic theorem:

\[ H = H(t) \]
\[ H(t = -\infty) = H_i \]
\[ H(t = +\infty) = H_f \]

Now break this path into N sections:

\[ H_i \rightarrow H_2 \rightarrow H_3 \rightarrow H_4 \rightarrow H_5 \rightarrow H_6 \rightarrow \cdots \rightarrow H_f \]

Now suppose enough pieces so that

\[ H_i \rightarrow H_{j+1} \] is a small perturbation

\[ H_{j+1} = H_j + H' \] with $H'$ small

\[ H(t) \approx H_j + \frac{t-t_j}{4\varepsilon} H' \] higher derivatives negligible

Now time evolution is via pert theory

in going from $t_j \rightarrow t_{j+1}$

in pert theory suppose at $t_j$ the system is in a given eigen state of

$H_j$ call it the $\alpha_i$ state
Now for each segment, ordinary pert theory is left.

For each segment compute $d\Psi$, via 1st order td pert.

And $d\Psi$, via 1st order time indep. pert.

Compare and see that ad. theorem works for each segment. The add up.
The $\langle \psi | H | \psi \rangle$ is smallest for

$$H \psi = E \psi$$

to look for which $E$, try to find

$$\langle \psi | H | \psi \rangle = 0$$

Often we are not interested in finding $E$ exactly

as a system and we can set $\varepsilon_i$ to 0.
get exact answer or in a limited trial space and get an approximate answer.

Consider a variational family of normalized states parameterized by some set of parameters $\phi$, parameter family $|\psi(\alpha)\rangle$

where

$$|x| \langle |\psi(\alpha)\rangle = \sqrt{\frac{2}{\pi \alpha}} \exp \left( -\frac{x^2}{\alpha^2} \right)$$

Clearly this family of states does not span the entire space but we want to find the state in this family which best approximates the true ground state.

impose some variational condition on limited family
\[ \langle \psi(0) | \hat{H} | \psi(0) \rangle = 0 \]

by finding a value of \( \alpha \) which minimizes \( \langle \psi(0) | \hat{H} | \psi(0) \rangle \) that gives "best" approx to true wave function in limited class.

formally

\[ \frac{\partial \langle \psi(0) | \hat{H} | \psi(0) \rangle}{\partial \alpha} = 0 \]

gives equation for \( \alpha \), call solution \( \alpha_0 \).

\[ \psi(0) \] is approx wave function

\[ E_{\text{approx}} = \langle \psi_{\text{approx}} | \hat{H} | \psi_{\text{approx}} \rangle \]

How good an approx is it?

well it depends on how close my family of states is true state...