QUANTUM PHYSICS I

PROBLEM SET 6

SOLUTION

A. Eigenfunctions and eigenvalues of common operators

What are the eigenfunction and eigenvalues of the operators

i) $\hat{x}$

As seen in class, the eigenfunctions are $f_{x_0}(x) = \delta(x - x_0)$ with eigenvalues $x_0$, for any real $x_0$. In fact,

$$\hat{x} f_{x_0}(x) = \hat{x} \delta(x - x_0) = x \delta(x - x_0) = x_0 \delta(x - x_0) = x_0 f_{x_0}(x). \quad (1)$$

ii) $\hat{p}$

As seen in class, the eigenfunctions are $f_k(x) = e^{ikx}/\sqrt{2\pi}$ with eigenvalues $\hbar k$, for any real $k$. In fact,

$$\hat{p} f_k(x) = -i\hbar \frac{d}{dx} \frac{e^{ikx}}{\sqrt{2\pi}} = \hbar k \frac{e^{ikx}}{\sqrt{2\pi}} = \hbar k f_k(x). \quad (2)$$

Repeat items i) and ii).

B. Eigenfunctions of kinetic energy

What are the eigenfunctions and eigenvalues of the kinetic operator $\hat{K} = \hat{p}^2/2m$. Show two degenerate eigenfunctions of the kinetic operator which are orthogonal to each other. Also, show two degenerate eigenfunctions that are NOT orthogonal.

The eigenfunctions of $\hat{K}$ are the same as the ones of $\hat{p}$:

$$\hat{K} f_k(x) = \frac{1}{2m} \hat{p} \hat{p} f_k(x) = \frac{1}{2m} \hat{p} \hbar k f_k(x) = \frac{1}{2m} (\hbar k)^2 f_k(x), \quad (3)$$

and the corresponding eigenvalues are $\hbar^2 k^2/2m$. The functions $f_k(x)$ and $f_{-k}(x)$ are two linearly independent eigenfunctions (as long as $k \neq 0$) since they share the same eigenvalue $\hbar^2 k^2/2m$. They are also orthogonal since

$$\int_{-\infty}^{\infty} dx \ (f_{-k}(x))^* f_k(x) = \int_{-\infty}^{\infty} dx \ \frac{1}{2\pi} e^{ikx} e^{-ikx} = \delta(2k) = 0, \quad (4)$$

where we assumed that $k \neq 0$. An example of two non-orthogonal degenerate eigenfunctions is $f_k(x)$ and $17f_k(x) - 3if_{-k}(x)/\sqrt{17^2 + 3^2}$. 

C. Schroedinger equation in momentum space

Denote by \( |k\rangle \) the momentum eigenfunction with eigenvalue \( p = \hbar k \); that is

\[
\hat{p}|k\rangle = \hbar k|k\rangle,
\]
and by \( |n\rangle \) the energy eigenfunction of the hamiltonian \( \hat{H} = \hat{p}^2/2m + \hat{V} \) with eigenvalue \( E_n \)

\[
\hat{H}|n\rangle = E_n|n\rangle.
\]

Write the time independent Schroedinger equation in \( 6 \) in the basis of momentum eigenfunctions. You should obtain an equation for \( \psi_n(k) = \langle k|n\rangle \) depending on the “matrix element” \( \langle k|\hat{V}|k'\rangle \). Hint: the answer is

\[
\psi_n(k) \left[ \frac{\hbar^2 k^2}{2m} - E_n \right] + \int dk' \langle k|\hat{V}|k'\rangle \psi_n(k') = 0.
\]

Multiplying the Schrödinger on the left by \( \langle k| \) and inserting the identity \( \mathcal{P} = \int dk'|k'\rangle\langle k'| \) we have

\[
\langle k|\hat{H} \int dk' \langle k'|k'\rangle \langle k'|n\rangle = \langle k|E_n|n\rangle
\]

\[
\int dk' \langle k| \frac{\hat{p}^2}{2m} + \hat{V}(\hat{x})|k'\rangle \langle k'|n\rangle = E_n \langle k|n\rangle
\]

\[
\int dk' \left[ \frac{k^2}{2m} \langle k|k'\rangle + \langle k|\hat{V}(\hat{x})|k'\rangle \right] \langle k'|n\rangle = E_n \langle k|n\rangle.
\]

Defining \( \psi_n(k) = \langle k|n\rangle \) we then have

\[
\psi_n(k) \left[ \frac{\hbar^2 k^2}{2m} - E_n \right] + \int dk' \langle k|\hat{V}|k'\rangle \psi_n(k') = 0.
\]

D. Probabilistic interpretation

A particle lives inside two impenetrable walls at \( x = 0 \) and \( x = a \). Its wave function at time \( t \) is given by

\[
\Psi(x, t) = \begin{cases} 
\frac{\sqrt{2}}{a} x, & \text{for } 0 < x < a, \\
0, & \text{otherwise}
\end{cases}
\]

(10)

i) If the energy is measured, what are the possible outcomes and with which probabilities?
First, let us normalize the wave function properly:

\[ 1 = \int_{0}^{a} A^2 x^2 = \frac{a^3 A^2}{3} \rightarrow A = \sqrt{\frac{3}{a^3}}. \]  \hfill (11)

The possible outcomes of an energy measurement are the eigenvalues of the infinite square well Hamiltonian, namely

\[ E_n = \frac{\hbar^2 \pi^2}{2ma^2} n^2, n = 1, 2, \ldots \]  \hfill (12)

To find the probabilities of each one we write the wave function as a linear superposition of eigenfunctions of the Hamiltonian

\[ Ax = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}} \sin(n \pi x / a). \]  \hfill (13)

The values of the constants \( c_n \) are given by

\[ c_n = \int_{0}^{a} \sqrt{\frac{3}{a^3}} x \sqrt{\frac{2}{a}} \sin(n \pi x / a) = -\sqrt{6} \frac{(-1)^n}{n \pi}. \]  \hfill (14)

The probability of finding the value \( E_n \) is given by \( |c_n|^2 = 6/(n^2 \pi^2) \).

ii) Suppose the energy measurement results in the value \( E = 4\hbar^2 \pi^2 / (2ma^2) \). What is the expected (average) value of the position immediately after this measurement?

If the energy value \( E_2 \) is measured, the wave function collapses to \( \psi_2(x) = \sqrt{\frac{2}{a}} \sin(2 \pi x / a) \).

A subsequent measurement of the position will yield the value \( x \) with probability \( |\sqrt{\frac{2}{a}} \sin(2 \pi x / a)|^2 \). The expected value will be

\[ \int_{0}^{a} dx \ x \frac{2}{a} \sin^2(2 \pi x / a) = \frac{a}{2}. \]  \hfill (15)