QUANTUM PHYSICS I SUMMARY  
(UP TO 1ST MIDTERM)

GENERAL FORMALISM

• state of the particle at a particular time described by a complex function of the position: the wave function \( \Psi(x) \).

• every classical observable corresponds to an operator

  position: \( x \rightarrow \hat{x} \), \( \hat{x}(\Psi(x)) = x(\Psi(x)) \)  
  momentum: \( p \rightarrow \hat{p} \), \( \hat{p}(\Psi(x)) = -i\hbar \frac{\partial}{\partial x} \Psi(x) \)  
  kinetic energy: \( T = \frac{p^2}{2M} \rightarrow \hat{T} = \frac{\hat{p}^2}{2M} \), \( \hat{T}(\Psi(x)) = -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} \Psi(x) \)  
  total energy (Hamiltonian): \( H = \frac{p^2}{2M} + V(x) \rightarrow \hat{H} = \frac{\hat{p}^2}{2M} + V(\hat{x}) \), \( \hat{H}(\Psi(x)) = \left[ -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + V(x) \right] \Psi(x) \)

• these operators have a set of eigenvalue/eigenfunction pairs

  \( \hat{A} \Psi_n(x) = a_n \Psi_n(x) \)  
  (sometimes \( n \)'s form a discrete set, sometimes a continuous set)

Examples:

  momentum: \( \hat{p} \Psi_p(x) = p \Psi_p(x) \rightarrow \Psi_p(x) = \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}, p = \text{real number} \)

  position: \( \hat{x} \Psi_{x_0}(x) = x_0 \Psi_{x_0}(x) \rightarrow \Psi_{x_0}(x) = \delta(x-x_0), x_0 = \text{real number} \)

  hamiltonian for free particle: \( \hat{H} \Psi_k(x) = E_k \Psi_k(x) \rightarrow \Psi_k(x) = \frac{e^{ikx}}{\sqrt{2\pi\hbar}}, E_k = \frac{k^2\hbar^2}{2M} \) for any real \( k \)
Hamiltonian of the infinite square well:

\[ H \psi_n(x) = E_n \psi_n(x) \quad \Rightarrow \quad \psi_n(x) = \frac{1}{\sqrt{L}} \sin \left( \frac{n \pi x}{L} \right), \quad E_n = \frac{n^2 \hbar^2}{2ML^2} \quad \text{for } n=1, 2, \ldots \]

In all these examples, the eigenfunctions satisfy an orthonormality condition:

**Discrete spectrum**

\[ \int_{-\infty}^{\infty} \psi_m^*(x) \psi_n(x) \, dx = \delta_{mn} = \begin{cases} 1 & \text{for } n=m \\ 0 & \text{for } n \neq m \end{cases} \]

Or

**Continuous spectrum**

\[ \int_{-\infty}^{\infty} \psi^+_p(x) \psi_p(x) \, dx = \delta(p-p') \]

- The wave function evolves in time according to the (time-dependent) Schrödinger equation:

\[ i \hbar \frac{\partial}{\partial t} \Psi(x,t) = -\frac{\hbar^2}{2M} \frac{\partial^2}{\partial x^2} \Psi(x,t) + V(x) \Psi(x,t) \]

The general solution of the Schrödinger eq. is given by

\[ \Psi(x,t) = \sum_n c_n e^{-iE_n t/\hbar} \psi_n(x) \]

where \( E_n, \psi_n(x) \) are the eigenvalues/eigenfunctions of \( \hat{H} \)

\[ \hat{H} \psi_n(x) = E_n \psi_n(x) \]

The \( c_n \)'s are determined by the initial condition

\[ \Psi(x,0) = \sum_n c_n \psi_n(x) \quad \Rightarrow \quad c_n = \int_{-\infty}^{\infty} \psi_n^*(x) \Psi(x,0) \, dx \]
• The only outcomes of a measurement of an observable $A(x)$ at time $t$ are the eigenvalues of $\hat{A}$. The probability of getting one particular eigenvalue $\lambda_n$ is given by $\lambda_n |c_n|^2$ where $c_n$ is the coefficient of the expansion of $\Psi$ in terms of $\phi_n$'s.

$$\hat{A} \phi_n(x) = \lambda_n \phi_n(x)$$

**Probability of getting this eigenvalue is the magnitude square of this**

$$\Psi(x,t) = \sum c_n \phi_n(x) \quad \Rightarrow \quad c_n = \int_0^\infty dx \phi_n^*(x) \Psi(x,t)$$

**Examples:**

1) Measurement of energy on an infinite square well when the wave function is

$$\Psi(x,t) = \frac{3}{\sqrt{5}} \frac{2}{\sqrt{L}} \sin \frac{2 \pi x}{L} - \frac{i}{\sqrt{5}} \frac{2}{\sqrt{L}} \sin \frac{4 \pi x}{L}$$

$$\phi_1(x), \phi_2(x), \phi_3(x)$$

The probability for $E_1 = \frac{\hbar^2}{2M L^2}$

$$|c_1|^2 = \frac{3}{5}$$

The probability for $E_3 = \frac{\hbar^2}{2M L^2}$

$$|c_3|^2 = \frac{2}{5}$$

Any other $E_n = |c_n|^2 = 0 \quad (n \neq 1, 3)$
2) measurement of momentum

when the wave function
is \( \Psi(x,t) = \psi(x) \)

\[
\psi(x) = \int_{-\infty}^{\infty} dp \frac{e^{i\frac{p-x}{\hbar}}} {\sqrt{2\pi\hbar}} c(p) \rightarrow c(p) = \int_{-\infty}^{\infty} dx \frac{e^{-i\frac{p-x}{\hbar}}} {\sqrt{2\pi\hbar}} \psi(x) = \text{"Fourier Transform of } \psi(x) \text{"}
\]

\( \psi(x) \) written as a linear combination of momentum eigenfunctions

probability (density) of measuring \( p \) is

\[
|c(p)|^2 = \left| \int_{-\infty}^{\infty} dx \frac{e^{-i\frac{p-x}{\hbar}}} {\sqrt{2\pi\hbar}} \psi(x) \right|^2
\]

3) measurement of position

when the wave function
is \( \Psi(x,t) = \phi(x) \)

\[
\phi(x) = \int_{-\infty}^{\infty} dy \phi(y) \delta(x-y) \rightarrow \phi(y) = \int_{-\infty}^{\infty} dx \delta(x-y) \psi(x) = \psi(y)
\]

\( \psi(x) \) written as a linear combination of position eigenfunctions

probability (density) of finding the particle at \( y \)

\[
|c(y)|^2 = |\phi(y)|^2
\]

It follows from the rule above that the average value of the measurement of \( A \) is given by

\[
\langle A \rangle = \int_{-\infty}^{\infty} dx \Psi^*(x,t) A \Psi(x,t)
\]