Numerical Solution to Harmonic Oscillator

Notice that the probability density $\psi^2$ is confined to the bottom of the well, decays into classically forbidden region, and acquires more nodes (where $\psi^2 = 0$) as $n$ increases.
Ground State properties from Heisenberg uncertainty reln

Energy of a wavefunction is determined by the expectation value of Hamiltonian for Harmonic Oscillator $\langle H \rangle = \frac{\langle p^2 \rangle}{2m} + \frac{1}{2} m \omega^2 \langle x^2 \rangle$.

Since $\langle p \rangle = 0$ for bound state and $\langle x \rangle = 0$ for symmetric potential,

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \rightarrow \langle x^2 \rangle = \Delta x^2$$
$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \rightarrow \langle p^2 \rangle = \Delta p^2$$

So $\langle H \rangle = \frac{\Delta p^2}{2m} + \frac{1}{2} m \omega^2 \Delta x^2$ but we don't know either $\Delta x$ or $\Delta p$.

However, we know the two are connected by the Heisenberg reln:

$$\Delta x \Delta p \geq \frac{\hbar}{2}.$$ Using $\Delta p \geq \frac{\hbar}{2 \Delta x}$, we have

$$\langle H \rangle \leq \frac{\hbar^2}{8m\Delta x^2} + \frac{1}{2} m \omega^2 \Delta x^2$$

The ground-state energy must be lower than the minimum value of the RHS.
Minimization

\[
\frac{d}{d\Delta x} \left( \frac{\hbar^2}{8m \Delta x^2} + \frac{1}{2} m \omega^2 \Delta x^2 \right) = \frac{-\hbar^2}{4m \Delta x^3} + m \omega^2 \Delta x = 0 \quad \text{which has sol'n } \Delta x = \sqrt{\frac{\hbar}{2m\omega}}
\]

Let's compare to the exact solution \( \psi_0 \propto e^{-\frac{m\omega x^2}{2\hbar}} \).

Probability density \( \psi^* \psi_0 \propto e^{-\frac{m\omega x^2}{2\hbar}} = e^{-\frac{x^2}{\Delta x^2}} \) where \( \Delta x = \sqrt{\frac{\hbar}{2m\omega}} \).

We can also calculate the upper bound on energy of the ground state:

\[
\langle \hat{H} \rangle \leq \frac{\hbar^2}{8m \Delta x^2} + \frac{1}{2} m \omega^2 \Delta x^2 = \frac{\hbar^2}{8m \frac{\hbar}{2m\omega}} + \frac{1}{2} m \omega^2 \frac{\hbar}{2m\omega} = \frac{\hbar^2}{4} + \frac{\hbar^2}{4} = \frac{\hbar^2}{2}
\]

Same as analytical value for \( E_0 \)!

In this case, the inequality is an equality because the gaussian ground state minimizes the product \( \Delta x \Delta p \).

In general, of course, ground state wavefunctions are not gaussian and this procedure only gives an upper bound.