Harmonic Oscillator

\[ H = \frac{p^2}{2m} + \frac{1}{2} mw^2 x^2 = \frac{1}{2m} \left( p^2 + (mw x)^2 \right) \]

Try to factor \( H \): 

\[ H \equiv \frac{1}{2m} \left( i p + mw x \right) \left( -i p + mw x \right) = \frac{1}{2m} \left( p^2 + (mw x)^2 + ip mw x + mw x (-ip) \right) \]

\[ = \frac{1}{2m} \left( p^2 + (mw x)^2 + im \omega \left( p x - xp \right) \right) \]

Note that \([p, x] = \frac{\hbar}{i} (d/dx(x \psi) - x (d/dx) \psi) = \frac{\hbar}{i} (\psi + x \frac{d \psi}{dx} - x \frac{d \psi}{dx}) = \frac{\hbar}{i} \psi\]

So \([p, x] = \frac{\hbar}{i} = -i\hbar \quad \text{and} \]

\[ H = \frac{1}{2m} \left[ (ip + mw x)(-ip + mw x) - m \hbar^2 \right] \]

\[ H = \hbar \omega \left[ \frac{1}{\sqrt{2m\hbar \omega}} (ip + mw x) \frac{1}{\sqrt{2m\hbar \omega}} (-ip + mw x) - \frac{1}{2} \right] = \hbar \omega \left( a_+ a_- - \frac{1}{2} \right) \]
Ladder operators

\[ a_- a_+ = \frac{\hbar}{2m} + \frac{1}{2} \]

\[ a_+ a_- = \frac{\hbar}{2m} - \frac{1}{2} \quad (\text{since} \ [p,x]=-[x,p]) \]

\[ [a_-, a_+] = 1 \]

If \( \psi \) is an eigenfunction of \( H \), \( H\psi = \varepsilon \psi \). Then, what is \( a_+ \psi = ? \)

\[ H(a_+ \psi) = \hbar \omega (a_+ a_- + \frac{1}{2}) (a_+ \psi) = \hbar \omega (a_+ a_- a_+ + \frac{1}{2} a_+) \psi \]

\[ = a_+ \hbar \omega (a_- a_+ + \frac{1}{2}) \psi = a_+ \hbar \omega \left( \frac{\hbar}{2m} + \frac{1}{2} + \frac{1}{2} \right) \psi \]

\[ = a_+ \left( \varepsilon + \hbar \omega \right) \psi = (\varepsilon + \hbar \omega) (a_+ \psi) \]

So \( a_+ \psi \) is also an eigenfunction of \( H \), with eigenvalue greater by \( \hbar \omega \)! We therefore call \( a_+ \) the "raising" operator.
Likewise,
\[ H(a_- \psi) = \hbar \omega (a_- a_+ - \frac{1}{2})(a_- \psi) = \hbar \omega (a_- a_+ a_- - \frac{1}{2} a_-) \psi \]
\[ = a_- \hbar \omega (a_+ a_- - \frac{1}{2}) \psi = a_- \hbar \omega (\frac{\hbar}{\hbar \omega} - \frac{1}{2} - \frac{1}{2}) \psi \]
\[ = a_- (H - \hbar \omega) \psi = a_- (\varepsilon - \hbar \omega) \psi = (\varepsilon - \hbar \omega)(a_- \psi) \]

So \( a_- \psi \) is also an eigenfunction of \( H \), with eigenvalue reduced by \( \hbar \omega \). We therefore call \( a_- \) the "slow mig" operator.
Eventually, $\alpha - \Psi_0 = 0$

$$\frac{1}{\sqrt{2\pi m\hbar}} \left( ip + mw x \right) \Psi_0 = 0$$

$$\left( \frac{\hbar}{i} \frac{d}{dx} + mw x \right) \Psi_0 = 0$$

$$\frac{d}{dx} \Psi_0 = -\frac{mw x}{\hbar} \Psi_0$$

This is a 1st-order differential eqn we can use to find $\Psi_0$:

$$\int \frac{d\Psi_0}{\Psi_0} = -\int \frac{mw x}{\hbar} dx$$

$$\ln \Psi_0 = -\frac{mw x^2}{2\hbar} + C$$

$$\Psi_0 = A e^{-\frac{mw x^2}{2\hbar}}$$

**Normalization:**

$$\int_{-\infty}^{\infty} \Psi_0^* \Psi_0 dx = \int_{-\infty}^{\infty} A^2 e^{-\frac{mw x^2}{\hbar}} dx$$

$$= A^2 \sqrt{\frac{2\pi \hbar}{mw}} = 1$$

$$\Rightarrow A = \left( \frac{mw}{\pi \hbar} \right)^{1/4}$$
Generate $\Psi_1$

$\Psi_0 = \left( \frac{m \omega}{\pi \hbar} \right)^{1/4} e^{-\frac{m \omega x^2}{2\hbar}}$

$\alpha_t = \frac{1}{\sqrt{2m \hbar}} (-i \rho + m \omega x)$

$\Psi_1 = \alpha_+ \Psi_0 = \frac{1}{\sqrt{2m \hbar}} \left( -\frac{k}{\sqrt{2}} + m \omega x \right) \left( \frac{m \omega}{\pi \hbar} \right)^{1/4} e^{-\frac{m \omega x^2}{2\hbar}}$

$= \left( \frac{1}{\sqrt{\frac{m \omega \pi \hbar^3}{3}}} \right)^{1/4} \left( \frac{k}{\sqrt{2}} \frac{m \omega}{\hbar} x + m \omega x \right) e^{-\frac{m \omega x^2}{2\hbar}}$

$= \left[ \left( \frac{m \omega}{\pi \hbar} \right)^{3/4} \frac{4}{3} \right]^{1/4} x e^{-\frac{m \omega x^2}{2\hbar}}$

Note:

$\int_{-\infty}^{\infty} \Psi_1^* \Psi_1 \, dx = \frac{2}{\sqrt{\pi}} \left( \frac{m \omega}{\pi \hbar} \right)^{3/2} \int_{-\infty}^{\infty} x^2 e^{-\frac{m \omega x^2}{2\hbar}} \, dx = \frac{2}{\sqrt{\pi}} \gamma \int_{-\infty}^{\infty} x^2 e^{-\gamma x^2} \, dx = \frac{2}{\sqrt{\pi}} \gamma^{3/2} \left[ \frac{\pi^{3/2}}{2 \gamma^{3/2}} \right] = 1$

($\gamma = \frac{m \omega}{\pi \hbar}$)

However, higher $\Psi_n$ are not automatically normalized...
Spectrum

Since \( a_- |\psi_0 \rangle = 0 \), \( \hat{H} |\psi_0 \rangle = \hbar \omega (a_+ a_- + \frac{1}{2}) |\psi_0 \rangle = \frac{\hbar \omega}{2} |\psi_0 \rangle \).

Higher eigenvalues are spaced by \( \hbar \omega \), so we have

\[
E_n = (n + \frac{1}{2}) \hbar \omega, \quad n = 0, 1, 2, \ldots
\]

(This is equivalent to recognizing that the eigenvalues of \( a_+ a_- \) are \( n = 0, 1, 2, \ldots \).)

\[\begin{array}{c}
\text{energy} \\
\hline
\text{V(x)} \\
\hline
\text{Classical points} E = V(x)
\end{array}\]

"ground state"