1. Consider an electron bound to a two-dimensional infinite quantum well with sides of length $x = \sqrt{8}L$ and $y = \sqrt{3}L$.
   
a. Write down the time-independent differential wave equation governing the energy of this system. (1)
   
b. Solve this equation for the stationary-state wavefunctions $\Psi(x, y)$, and determine all the allowed energies, using quantum numbers $n_x$ and $n_y$. What is the lowest “ground-state” energy? (2)
   
c. Calculate the energies for the next three higher energy levels. For each distinct energy value, list the possible combinations of $n_x$ and $n_y$ and the degree of degeneracy. (3)

\[
\begin{align*}
-\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Psi(x, y) &= E \Psi(x, y) \quad \text{for} \quad 0 < x < \sqrt{8}L \quad 0 < y < \sqrt{3}L \\
\end{align*}
\]

By separation of variables, $\Psi(x, y) = \Psi_x(x) \Psi_y(y)$ leads to two ordinary differential eqns

\[
\begin{align*}
-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \Psi_x(x) &= E_x \Psi_x(x) \\
-\frac{\hbar^2}{2m} \frac{d^2}{dy^2} \Psi_y(y) &= E_y \Psi_y(y) \\
\end{align*}
\]

Applying b.c.'s $\Psi_x(x=0, \sqrt{8}L) = 0$ and $\Psi_y(y=0, \sqrt{3}L) = 0$ gives eigenvalues s.t.,

\[
E = \frac{\hbar^2}{2m} \left( \frac{\pi^2 n_x^2}{8L^2} + \frac{\pi^2 n_y^2}{3L^2} \right)
\]

$\Psi(x, y) = A \sin(K_n x) \sin(K_n y)$ where $A = \sqrt{\frac{2}{\sqrt{8}L}} \sqrt{\frac{2}{\sqrt{3}L}}$

Ground state ($n_x = n_y = 1$) has energy $E = \frac{\hbar^2}{2m} \left( \frac{1}{8} + \frac{1}{3} \right) = \frac{5\hbar^2}{2mL^2} \frac{11}{24}$

<table>
<thead>
<tr>
<th>$n_x$</th>
<th>$n_y$</th>
<th>$E \left( \frac{\hbar^2 n_x^2}{2mL^2} + \frac{\hbar^2 n_y^2}{3L^2} \right)$</th>
<th>degeneracy</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>$\frac{1}{8} + \frac{1}{3} = \frac{11}{24}$</td>
<td>1</td>
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<tr>
<td>1</td>
<td>2</td>
<td>$\frac{1}{8} + \frac{1}{2} = \frac{5}{8}$</td>
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<tr>
<td>3</td>
<td>1</td>
<td>$\frac{1}{8} + \frac{1}{3} = \frac{5}{8}$</td>
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<tr>
<td>2</td>
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<td>$\frac{1}{8} + \frac{1}{3} = \frac{11}{24}$</td>
<td>1</td>
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</tbody>
</table>
2. The angular part of the wavefunction for an electron bound in a hydrogen atom is \( \psi(\theta, \phi) = C(5Y^3_4 + Y^3_6 + Y^0_6) \), where \( Y^m_l(\theta, \phi) \) are the normalized spherical harmonics.
   
a. What is the value of normalization constant \( C? \) (1)
   
b. What is the probability of finding the atom in a state with \( m=3? \) (2)
   
c. What are the expectation values of angular momentum operators \( L_z \) and \( L^2? \) (4)

\[ \int \psi^* \psi \, d\Omega = 1 \]  
Since \( Y^m_l \) is normalized, 
\[ C^2 \cdot (5^2 + 1^2 + 1^2) = 1 \]
So, 
\[ C = \frac{1}{\sqrt{27}} \]

\[ \frac{5^2 + 1^2}{27} = \frac{26}{27} \]

\( Y^m_l \) are eigenstates of \( L_z \) and \( L^2 \) with eigenvalues \( \pm m \) and \( \hbar^2 l(l+1) \).

Therefore, 
\[ \langle L_z \rangle = \frac{25}{27} \cdot 3 + \frac{1}{27} \cdot 3 + \frac{1}{27} \cdot 0 = \frac{24}{27} \cdot 3 \hbar \]
\[ \langle L^2 \rangle = \frac{25}{27} \cdot \hbar^2 \cdot \frac{4}{4+1} + \frac{1}{27} \cdot \hbar^2 \cdot \frac{6}{6+1} + \frac{1}{27} \cdot \hbar^2 \cdot \frac{6}{6+1} \]

\[ = \frac{58}{27} \left( 25 \cdot 20 + 42 + 42 \right) = \frac{58 \cdot 1}{27} \hbar^2 \]
3. The wavefunction $\psi(x) = Ae^{-b^2 x^2}$ is the ground state of a one-dimensional harmonic oscillator. $A$ and $b$ are real constants.

   a. What are the units of $A$ and $b$? (2)
   b. Normalize the wavefunction to determine the value of $A$ (assume it is real). (2)
   c. What is the potential $V(x)$ in terms of $\hbar$, $m$, and $b$? (2)
   d. What is the energy of this state? What is the energy of the first excited state? (2)

   a. Since $\int \psi^* \psi \, dx = 1$, $A$ must be units of $\text{length}^{-1/2}$.
   Since $b^2 x^2$ is unitless, $b$ has units of $\text{length}^{-1}$.

   b. $\int \psi^* \psi \, dx = A^2 \int e^{-2b^2 x^2} \, dx = A^2 \sqrt{\frac{\pi}{2b^2}} = 1 \quad \text{so} \quad A = \left( \frac{2}{\pi} \right)^{1/4} b^{1/2}$

   c. For $V(x) = \frac{1}{2} m \omega^2 x^2$, ground state wavefunction is $\propto e^{-\frac{m \omega}{2a} x^2}$
   Therefore, $b^2 = \frac{m \omega}{2a}$ and $\omega = \frac{2b^2 m}{\hbar}$ so that
   $V(x) = \frac{1}{2} m \left( \frac{2b^2 m}{\hbar} \right) x^2 = \frac{2}{\hbar} \frac{b^2 m^2}{\hbar^2} x^2$

   d. $E = \hbar \omega \left( n + \frac{1}{2} \right)$, $n = 0, 1, 2, \ldots$ so ground state has $E = \frac{\hbar \omega}{2} = \frac{\hbar^2 b^2}{m}$
   1st excited state ($n = 1$) has $E = \frac{3\hbar \omega}{2} = \frac{3\hbar^2 b^2}{2m}$
4. Protons have spin ½, just like electrons. However, their mass is approximately 1 GeV/c², nearly 2000 times larger. Decide whether their associated magnetic moment is (greater than/less than/equal to) the electron magnetic moment $\mu_B$, and explain why. (3)

\[
\vec{\mathcal{A}} = \text{current area} = -e f \cdot \pi r^2 = e \frac{V}{2}\pi r^2 = -\frac{e V r}{2} = \frac{e (m V r)}{2m} = \frac{(\text{e}h) \frac{L}{\text{h}}}{(2m) \frac{\text{h}}{4}} = \frac{\mu_B}{\frac{1}{4}}
\]

Since $\mu_B \propto \frac{1}{m}$, the proton magnetic moment will be $\approx 2000$ smaller!
5. Consider a three-level system where the Hamiltonian and observable $A$ are given by the matrix
\begin{align*}
\hat{H} &= \hbar \omega \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad \hat{A} = \mu \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\end{align*}

a. What are the possible values obtained in a measurement of $A$? (2)

b. Does a state exist in which both the results of a measurement of energy $E$ and observable $A$ can be predicted with certainty? Why or why not? (2)

c. Two measurements of $A$ are carried out, separated in time by $t$. If the result of the first measurement is its largest possible value, determine the expectation value $\langle \psi(t) | A | \psi(t) \rangle$ for the second measurement. (3)

\[ \text{eigenvalues of } \hat{A} \text{ are roots of characteristic eqn} \]
\[ \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad -\lambda (\lambda^2 - 1) + 2\lambda = 0 \quad \Rightarrow \quad \lambda = 0, \pm \sqrt{2} \mu \]

\[ \text{No, because } \hat{H} \text{ and } \hat{A} \text{ do not have the same eigenvectors} \]

\[ \text{eigenvector associated with } \lambda = \pm \sqrt{2} \mu \text{ is} \]
\[ M \begin{bmatrix} 1 \\ -\sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad -\sqrt{2} a + b = 0 \quad a - \sqrt{2} b + c = 0 \quad b - \sqrt{2} c = 0 \quad \Rightarrow \quad \begin{cases} a = \frac{1}{2} \\ b = \frac{\sqrt{2}}{2} \\ c = \frac{1}{2} \end{cases} \]

Now decompose into eigenvectors of $\hat{H}$ and add time-dependent phase:
\[ \Phi(t) = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-i\omega t} + \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix} e^{i\omega t} \]

Then, $\langle \Phi(t) | \hat{A} | \Phi(t) \rangle = \frac{1}{2} \begin{bmatrix} e^{i\omega t} + \sqrt{2} e^{-i\omega t} \end{bmatrix} M \frac{1}{2} \begin{bmatrix} e^{i\omega t} \\ \sqrt{2} e^{-i\omega t} \end{bmatrix} = \frac{\sqrt{2}}{2} \cos \omega t$
6. Consider an electron incident from \( x = -\infty \) on the following one-dimensional potential:
\[
V(x) = \begin{cases} 
D\delta(x + a), & x < 0 \\
\infty, & x \geq 0
\end{cases}
\]

a. Draw the potential. (1)

b. What are the units of \( D \)? (1)

c. What are the boundary conditions on the wavefunction at \( x = -a \)? (2)

d. What is the absolute magnitude of the reflection coefficient, and why? (2)

\[ \int V(x) \, dx = \text{length, energy} \]  
\[ \int D\delta(x+a) \, dx = D \text{ has same units.} \]

B.C.s given by
\[
\int_{-a}^{a} \left[ -\frac{\hbar^2}{2m} \psi'' + D\delta(x+a)\psi \right] \, dx = E\psi \quad \text{(and \( \psi \) continuing)}
\]

\[
-\frac{\hbar^2}{2m} \psi' \bigg|_{-a}^{a} + D\psi(a) = 0 \implies A\psi' = \frac{2mD}{\hbar^2} \psi(a)
\]

\[ r = 1 \text{ because } \psi = 0 \text{ for } x > 0 \text{ so no transmitted probability flux.} \]
7. Consider the following angular part of the wavefunction for an electron in a hydrogen atom:
\[ \psi(\theta, \phi) = \frac{1}{\sqrt{3}} (Y^0_1 | \uparrow \rangle + \sqrt{2} Y^1_1 | \downarrow \rangle) \] where | \uparrow \rangle and | \downarrow \rangle are eigenstates of \( s_z \).

a. Without any knowledge of the radial part of the wavefunction, what is the **minimum** value that could be returned as the result of an energy measurement? (1)
b. With what probability will the result of a measurement of spin along \( z \) give \( +\hbar/2 \) ? (1)
c. The total angular momentum is the sum of orbital and spin: \( J=L+S \). Compute the expectation value of the operator \( \langle J_z \rangle \). (3)

\[ \begin{align*}
&\text{a.}\quad \text{Since } L=1, \text{ the minimum value of } n \text{ is } 2. \quad \text{Therefore } E = -\frac{\hbar^2}{2m} \langle \hat{L}_z \rangle = -\frac{\hbar^2}{2m} \langle \hat{L}_z \rangle = -13.6 eV \approx -3.4 eV \\
&\text{b.}\quad \frac{1}{2} \langle \hat{S}_z \rangle = \frac{1}{3} \\
&\text{c.}\quad \text{Hilbert Space is } L \otimes S (6 \times 6), \quad J_z = L_z \otimes 1 + S_z \otimes 1 \\
\text{Let } | L=1, m=0 \rangle \otimes | \uparrow \rangle + \frac{1}{\sqrt{3}} | L=1, m=1 \rangle \otimes | \downarrow \rangle = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 1 \end{bmatrix} \\
\text{So } \langle \psi | J_z | \psi \rangle = \\
\frac{1}{15} \begin{bmatrix} 0 & \sqrt{2} & 0 & 0 & 0 \\ \sqrt{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3 \sqrt{2} & 1 & 0 & 0 & 0 \\ 1 \sqrt{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 1 \end{bmatrix} = \frac{\hbar}{3} \frac{3}{2} = \frac{\hbar}{2} 
\end{align*} \]
8. Identify the quantum numbers ($l$ and $m$) for the following spherical harmonics found on the www: (4 total)
Potentially useful formulae:

\[
\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} (\hat{x} + \frac{i}{m\omega} \hat{p})
\]

\[
\hat{p} = -i \sqrt{\frac{m\omega\hbar}{2}} (\hat{a} - \hat{a}^+)
\]

\[
[\hat{a}, \hat{a}^+] = 1
\]

\[
|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^+)^n |0\rangle
\]

\[
\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) \exp\left(\frac{1}{2}\xi^2\right), \quad \text{where} \quad \xi = \sqrt{\frac{m\omega\hbar}{\hbar}} x
\]

\[
H_0 = 1 \quad H_1 = 2\xi \quad H_2 = 4\xi^2 - 2 \quad H_2 = 8\xi^3 - 12\xi
\]

\[
E_n = \hbar\omega \left(n + \frac{1}{2}\right)
\]

\[
\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}.
\]

\[
\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

\[
L^2 = \hbar^2 (\hat{\ell} + 1), \quad L_z = m\hbar.
\]