Particle in a box in Dirac notation: Dirac bra vector
energy eigen states: \( \{ \mid n \rangle \} \) and/or \( \{ \langle n \mid \} \) 
\( n \) Dirac ket vectors

Explicitly,
\[ \mid n \rangle = \text{column vector } = \begin{pmatrix} \cdots \mid n \rangle \end{pmatrix} \]

\[ \langle n \mid = \text{row vector } = (0, 0, 0, 0, \ldots) \]

The \( \langle n \mid n \rangle = \delta_{nn} \) are \( 1 \)s
\( < m \mid n > = \delta_{mn} \) orthogonality statements
in the Dirac notation.

An arbitrary state is written
\[ \text{arbitrary } = \langle n \mid = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \ddots \end{pmatrix} \]
where \( \{ a_n \} \) are the expansion coefficients.

\[ < n \mid n > = \sum_n a_n^* a_n = \sum_n |a_n|^2 = 1 \]

We can also write
\[ \text{arbitrary } = \mid a \rangle = \langle n \mid + q_2 \langle m \mid + q_3 \langle \ldots \mid + \cdots \]

We used to write this as \( \psi(x) = a_1 \phi_1(x) + a_2 \phi_2(x) + a_3 \phi_3(x) + \cdots \)
If we multiply an arbitrary state vector by \( |n\rangle \),
then we get

\[
\langle n|\text{arbitrary} \rangle = \langle n|a \rangle = \begin{pmatrix} \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ \vdots \end{pmatrix}
\]

\[
= a_n < a \text{ a complex #}
\]

\[
\Rightarrow \quad a_n = \langle n|a \rangle
\]

We used to write this as \( q_n = \int_{-\infty}^{\infty} \delta(x) \phi_n^*(x) \, dx \)

Now we can write

\[
|\text{arbitrary}\rangle = |a\rangle = a_1 |1\rangle + a_2 |2\rangle + a_3 |3\rangle + \cdots
\]

\[
= |1\rangle a_1 + |2\rangle a_2 + |3\rangle a_3 + \cdots
\]

\[
= |1\rangle \langle 1|a \rangle + |2\rangle \langle 2|a \rangle + |3\rangle \langle 3|a \rangle + \cdots
\]

\[
|a\rangle = \sum_n n\langle n|a \rangle
\]

In ordinary vectors this is

\[
\hat{a} = \hat{x} a_x + \hat{y} a_y + \hat{z} a_z
\]

\[
\hat{x} \hat{a} \quad \hat{y} \hat{a} \quad \hat{z} \hat{a}
\]
\[ a = \hat{x}(\hat{x} \cdot \hat{a}) + \gamma (\hat{x} \cdot \hat{a}) + \bar{z} (\hat{z} \cdot \hat{a}) \]

Let \[ \hat{x}_1 = \hat{x}, \]
\[ \hat{x}_2 = \gamma, \]
\[ \hat{x}_3 = \bar{z}. \]

Then \[ \hat{a} = \sum_{n=1}^{g} \hat{x}_n (\hat{x}_n \cdot \hat{a}) \]

Just \[ \lim_{n \to \infty} |a> = \sum_{n} |n><n|a> \]

But notice \[ |a> = \left[ \sum_{n} |n><n| \right] |a> \]

\[ \text{Factor out } |a> \]

This say \[ |a> = \left[ \text{thing in bracket} \right] |a> \text{ from every term} \]

\[ \therefore \left( \sum_{n} |n><n| \right) = \hat{I} = \text{Identity operator} \]

Mathematical statement of completeness for the \[ \{ |n> \} \]

Why is this useful? Suppose we want to multiply:

\[ \langle \text{arbitrary } a | \text{arbitrary } b \rangle = \langle a | b \rangle \]

\[ = \langle a | \hat{I} | b \rangle \]

\[ \uparrow \text{ insert Identity operator} \]

\[ = \delta_{ab}. \]
\[
\begin{align*}
&= \langle a_1 \mid \sum_n \langle n \rangle \langle n \mid b \rangle \\
&= \sum_n \langle a_1 \rangle \langle n b \rangle \\
&\downarrow \quad \text{Homework 5}
\{ q_n^* \}
&= \sum_n q_n^* b_n \\
&= (q_1^*, q_2^*, q_3^*, \ldots) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{pmatrix}
\end{align*}
\]

**Position basis.**

\( a_n = \text{AM amplitude to observe } E_n = \langle a \mid a \rangle \)

\( \chi(x) = \text{AM amplitude to observe } x = \langle \psi \mid a \rangle \)

Imagine that \( x \) is discrete:

\[ \begin{array}{c|cc}
\chi(x_1) & \chi(x_2) & \chi(x_3) \\
\hline
\chi(x_1) & 0 & \ldots & 0 \\
\chi(x_2) & 0 & 0 & 0 \\
\hline
\chi(x_3) & 0 & 0 & \ldots \\
\end{array} \]

The \( t(x) \) is like:

\[ t(x) \approx \begin{pmatrix} t_1 \\ t_2 \\ \vdots \end{pmatrix} \]

when \( t_1 = \chi(x_1) \)

\( \chi_2 = \chi(x_2) \)

\( \ldots \)
and  \( \Psi(x) \sim (\Psi_1^*, \Psi_2^*, \Psi_3^*, \ldots) \)

Thus  \( \langle \Psi | \Psi \rangle = \sum_n \Psi_n^* \Psi_n \)

In the continuum limit this is

\[
\langle \Psi | \Psi \rangle = \int \Psi^*(x) \Psi(x) \, dx
\]

Now \( \langle n | a \rangle = a_n \) picks out the amplitude to observe \( \Psi_n \).

What picks out the amplitude to observe \( \Psi \)?

\[
\langle ? | a \rangle = \Psi(x)
\]

\[\uparrow\]

What does this mean to get \( a_n \), we multiply by an energy eigenstate. To get \( \Psi(x) \), we must multiply by an eigenstate of \( x \). \( \langle x | \Psi \rangle \)

\[\uparrow\text{an eigenstate of position is bra vector form.}\]

\[
\therefore \quad \langle x | \Psi \rangle = \Psi(x)
\]

What do the \( |x\rangle \) look like?
In vector form,
\[ |x\rangle \sim \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ at the } x^{th} \text{ position}
\]

but \( x \) is actually continuous, so I can't really write it this way.

Nevertheless, \( \langle x | \psi \rangle = \psi(x) \leq \text{amplitude to observe } x \)
\( \langle a | \psi \rangle = \text{an } \in \text{-amplitude to observe } E_a \).

For \( |n\rangle \) we had \( \sum_n \langle n | x \rangle \langle x | 1 \rangle = 1 < \text{Completeness} \)
\( \langle x | \rangle \text{ are also complete, so } \int_0^\infty |x\rangle \langle x| \ dx = 1 \)

\( |x\rangle \) are continuous!

Must integrate within the same

We can use \( \hat{1} \) to calculate dot products.

\[ \langle a | b \rangle = \langle a | \hat{1} | b \rangle = \langle a | \int_0^\infty | x \rangle \langle x | b \rangle \ dx | b \rangle \]
\[ = \int \langle a | x \rangle \langle x | b \rangle \ dx \]

\( \hat{\psi}_a(x) \)
\[ \langle a|b \rangle = \int_{-\infty}^{\infty} \psi_a^*(x) \psi_b(x) \, dx \leq \text{How to calculate } \langle a|b \rangle \text{ in position space} \]

just like \[ \langle a|b \rangle = \sum_{n} q_n^* b_n \leq \text{How to calculate } \langle a|b \rangle = \text{energy space} \]
The momentum basis

For free particles, the energy eigenstates form a continuum. We use the momentum eigenstates, because they are complete and also energy eigenstates.

We write the momentum eigenstates in Dirac notation as $|k\rangle$.

What do these states look like? Well, in the position basis they are plane waves:

$$|k\rangle \sim e^{ikx}$$

In the position basis, they are a continuum of QM amplitudes.

We can use an equals sign if we project $|k\rangle$ onto $|x\rangle$:

$$\langle x|k\rangle = \frac{1}{\sqrt{2\pi}} e^{-ikx}$$

The $|k\rangle$ are complete for free particles:

$$\sum k |k\rangle\langle k| = I$$

So we can write an arbitrary state $|\Psi\rangle$ as

$$|\Psi\rangle = \sum_k \langle k|\Psi\rangle |k\rangle = \left( \int |k\rangle \langle k| \, dk \right) |\Psi\rangle = \int |k\rangle \langle k| \, dk \cdot |\Psi\rangle.$$ 

What is $\langle k|\Psi\rangle$? It is the arbitrary state as it appears in momentum space. In other words, it is the momentum space wave function $\phi(k)$. 
\[ \langle k | \psi \rangle = \phi(k) \] just like \[ \langle x | \psi \rangle = \psi(x) \].

\[ | \psi \rangle = \int |k\rangle \langle k| \phi(k) dk \] for continuous \(|k\rangle\).

(This is the analogous to
\[ | \psi \rangle = \sum_n a_n | n \rangle \] for discrete \(|n \rangle\)).

We can make this statement more concrete and easier to understand by projecting it into the position basis:

\[ \langle x | \psi \rangle = \langle x | \left( \int |k\rangle \phi(k) dk \right) \]

\[ = \int \langle x | k \rangle \phi(k) dk \]

\[ = \int \frac{i e^{ikx}}{\sqrt{2\pi}} \phi(k) dk \]

By projecting into the position basis, we receive the expression for the wavefunction as the Fourier Transform of \( \phi(k) \).

The \(|k\rangle\) are also orthonormal:

\[ \langle k | l \rangle = \delta(k-l) \] (delta function).

How can we demonstrate this? Use the position basis to calculate \( \langle k | l \rangle \):
\[ \langle k' | k' \rangle = \langle k | i \frac{\partial}{\partial x} 1 | k' \rangle = \langle k | \left( \int_{-\infty}^{\infty} i k' \langle x \rangle \langle x | 1 \rangle dx \right) | k' \rangle \]

\[
= \int_{-\infty}^{\infty} \langle k | \langle x \rangle \langle x | 1 \rangle dx \frac{1}{\sqrt{2\pi} \frac{\partial}{\partial x}} \frac{1}{\sqrt{2\pi} \frac{\partial}{\partial x}} \]

\[
= \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{ik'x} e^{-ikx} dx = \delta(k' - k).
\]

This is the delta function.

Similarly, for the position basis we have

\[ \int_{-\infty}^{\infty} \langle x | x \rangle dx = \hat{1} \text{ completeness} \]

\[ \langle k | x' \rangle = \delta(x - x') \text{ orthonormality} \]
Hermitean Operator

In QM we use three types of mathematical objects.

1. Complex numbers: \( C \)
2. State vectors (kets): \( |\text{arb}\rangle \)
3. Operators: \( \hat{A} \)

For (1) & (2), we have "partner" objects:

1. \( C \rightarrow C^* \), complex conjugation.
2. Bra vectors: \( \langle \text{arb}| \)

There is also a "partner" object for operators. It is called the "Hermitean Conjugate Operator", and written as \( \hat{A}^+ \). So we have:

<table>
<thead>
<tr>
<th>Mathematical object</th>
<th>&quot;Partner&quot;</th>
</tr>
</thead>
<tbody>
<tr>
<td>number: ( C )</td>
<td>( C^* )</td>
</tr>
<tr>
<td>vector: (</td>
<td>\text{arb}\rangle )</td>
</tr>
<tr>
<td>operator: ( \hat{A} )</td>
<td>( \hat{A}^+ )</td>
</tr>
</tbody>
</table>

These are the objects we use in QM.

We can write the laws of QM in terms of the "objects" or their "partners". They both describe the same thing. For example, the TISE is:

\[
\frac{i\hbar}{\partial t} \hat{\Psi}(x,t) = \hat{H} \hat{\Psi}(x,t)
\]
but we could also write

\[ -i \hbar \hat{\Psi}^* (x,t) = \frac{\hat{H} \hat{\Psi}^* (x,t)}{2} \]

this describes the same physics.

We will define \( \hat{A}^+ \), the meaning of \( \hat{A}^+ \) in a minute, but first let's discuss why it is important. Basically, when an operator is equal to its Hermitian conjugate, we say that the operator is "Hermitian".

\[ \text{If } \hat{A} = \hat{A}^* \text{, we say } \hat{A} \text{ is Hermitian}. \]

This situation is the moral equivalent of a real number, except for the case of an operator.

"Hermitian Operators" are important for two reasons.

1. They always have real eigenvalues. Since observable quantities must be real, we modify Postulate I:

Postulate I: For every observable \( \hat{A} \), there is a Hermitian operator \( \hat{A} \) for which \( \hat{A} |n\rangle = \alpha_n |n\rangle \)

In Direct Notation

Eigenvectors Eq.

2. Hermitian Operators have eigenfunctions, which are orthogonal.

\[ \text{IF } \{ |n\rangle \} \text{ are eigenvectors of } \hat{A}, \text{ and } \hat{A} = \hat{A}^* \text{, then } \langle m | n \rangle = \delta_{mn}. \]
The meaning of $A^+$

Loosely speaking, if $\hat{A}$ is an operator which can be applied to a ket-vector, then $A^+$ is the equivalent operator which should be applied to the partur bra-vector:

$$\hat{A} \langle \alpha | \beta \rangle \quad \Rightarrow \quad \langle \alpha | A^+ \beta \rangle$$

Strict definition: if $|\alpha\rangle$ and $|\beta\rangle$ are two ket vectors, then $\hat{A}$ and $\hat{A}^+$ can operate on them.

$$\hat{A}^+ |\beta\rangle \quad \text{is the operator which satisfies}$$

$$\langle \alpha | (\hat{A}^+) | \beta \rangle \quad \text{def} \quad \langle \alpha | \hat{A} | \beta \rangle$$

For all $|\alpha\rangle$ and $|\beta\rangle$.

Example 1 Let $\hat{D}$ be $\frac{d}{dx}$ is positon space.

What is $\hat{D}^+$?

Evaluate $\langle \alpha | (\hat{D} | \beta \rangle)$ in position space:

$$\langle \alpha | \left( \hat{D} \right) | \beta \rangle = \int_{-\infty}^{\infty} \frac{d\psi_k(x)}{dx} \psi_\beta(x) \, dx = \left( \frac{\hat{D}}{\psi_k(x)} \psi_\beta(x) \right)^\ast - \frac{\psi_k(x)}{\psi_\beta(x)} | \beta \rangle$$

Since $\psi_k(x) \psi_\beta(x)$ are normalizable.

$$\equiv \langle \alpha | (\hat{D}^+) | \beta \rangle$$

by definition of $\hat{D}^+$.
\[ \hat{D}^+ = \frac{d}{dx} \iff \hat{D} = \frac{d}{dx} \]

Since \( \hat{D}^+ \neq \hat{D} \), \( \hat{D} \) is not Hermitian, and it cannot represent an observable.

The momentum operator \( \hat{p} \) includes a factor of \( i \) so it will be Hermitian. \( \hat{p} = -i\hbar \frac{d}{dx} = \hat{p}^+ \).

**Example 2** If \( \hat{A} = \text{multiplication by a number } c \),

The \( \hat{A}^+ = \text{multiplication by } c^* \).

Proof:

\[ \langle \alpha | (\hat{A}^+ | \beta \rangle = \langle \alpha | (c | \beta \rangle = c \langle \alpha | c | \beta \rangle = c \int_{-\infty}^{\infty} \psi^*_\alpha(x) \psi_\beta(x) dx \]

\[ = \int_{-\infty}^{\infty} \langle \psi^*_\alpha(x) | \hat{A}^* | \psi_\beta(x) \rangle dx \]

\[ = \int_{-\infty}^{\infty} \langle \psi^*_\alpha(x) | \psi_\beta(x) \rangle \psi^*_\beta(x) \psi_\beta(x) dx \]

\[ = \int_{-\infty}^{\infty} \langle \psi^*_\alpha(x) | \psi_\beta(x) \rangle \psi^*_\beta(x) \psi_\beta(x) dx \]

\[ \therefore \hat{A}^+ = \text{multiplication by } c^* \]

Now we prove the two important consequences of an operator being Hermitian:

**Important Property**

1. If \( \hat{A} = \hat{A}^+ \), then the eigenvectors of \( \hat{A} \) are real; \( \hat{A} | \alpha \rangle = \alpha_n | \alpha \rangle \) where \( \alpha_n = \alpha_n^* \) and \( | \alpha \rangle = \text{eigenvector of } \hat{A} \)

Proof:

Evaluate \( \langle n | (\hat{A}^+ | \alpha \rangle = \langle n | \alpha_n \rangle = \alpha_n \langle n | \alpha \rangle \) by def of \( \hat{A}^+ \)

\[ \langle n | \hat{A}^+ | \alpha \rangle = \int \psi^*_\alpha(x) \hat{A}^* \psi_\alpha(x) dx = \int \alpha_n(x) \hat{A} \psi_\alpha(x) dx \]

\[ = \int \alpha_n(x) \psi_\alpha(x) dx = \alpha_n \int \psi_\alpha(x) dx = \alpha_n \langle n | \alpha \rangle \]

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\[ = \int \alpha_n(x) \psi_\alpha(x) dx = \alpha_n \int \psi_\alpha(x) dx = \alpha_n \langle n \alpha \rangle \]
\[ a_{m} = a_{n}^* \] and \( a_{n} = \text{real number} \).

**Proof**

**Important Property (2)**

If \( |m\rangle \) and \( |l\rangle \) are eigenvectors of \( \hat{A} \), then

\[ \langle m|n\rangle = \delta_{mn} \quad \text{the eigenvectors are orthogonal.} \]

**Proof:** \( \hat{A} |m\rangle = a_{m} |m\rangle \) and \( \langle n|\hat{A} = a_{n} \langle n| \) by (P1).

\[ \hat{A} |m\rangle = a_{m} |m\rangle \quad \text{and} \quad \langle m|\hat{A} = a_{m} \langle m| \]

Therefore

\[ \langle m|\hat{A}|n\rangle = \langle m|a_{n}|m\rangle = a_{n} \langle m|n\rangle \]

\[ \langle m|\hat{A}|n\rangle = a_{m} \langle m|n\rangle \]

\[ (a_{m} - a_{n}) \langle m|n\rangle = 0. \]

**Three cases**

1. \( m = n \). Then \( \langle m|\hat{A}|n\rangle = a_{n} \).
2. \( m \neq n \) and \( a_{m} \neq a_{n} \).
   - \( m = n \). Then \( \langle m|\hat{A}|n\rangle = a_{n} \)
   - If \( |m\rangle \) is normalized, \( \langle m|m\rangle = 1 \).
3. \( m \neq n \), but \( a_{m} = a_{n} \).

In this case two different states \( |m\rangle \) & \( |n\rangle \) have the same eigenvalue. We call this situation "degenerate eigenstates."
In the case of degenerate eigenstates, it turns out that we can make linear combinations of the states which are orthogonal. We'll ignore this detail and jump straight to the conclusion:

\[ |m \rangle \langle n| = \delta_{mn} \]

For getting the eigenstates of a Hermitian operator.