The simplest bound-state system in QM is the particle-in-a-box, and its mathematical description is a Fourier Series.

Let \( F(x) \) be (1) periodic with period \( 2L \) and (2) square integrable between \(-L \) and \( L \).

Then \( F(x) \) can be written as a Fourier Series:

\[
F(x) = \sum_{n=-\infty}^{\infty} c_n \cos \left( \frac{n \pi x}{L} \right) \] for some set of coefficients \( \{c_n\} \)

\( \uparrow \) This form can represent a real \( F(x) \) or complex \( F(x) \).

But if \( F(x) \) is real then we can also write the Fourier Series as

\[
F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n \pi x}{L} \right) + b_n \sin \left( \frac{n \pi x}{L} \right) \right]
\]

for some set of \( \{a_n\}, \{b_n\} \).

For real \( F(x) \), we can convert between the two forms like this:

\[
c_n = \frac{1}{2} (a_n - ib_n), \quad n > 0
\]

\[
\frac{1}{2} (a_0), \quad n = 0
\]

\[
\frac{1}{2} (a_n + ib_n), \quad n < 0
\]

\[
a_n = c_n + c_{-n}
\]

\[
b_n = i(c_n - c_{-n})
\]
To show that the two forms are equivalent, just substitute for $e^x$ and use Euler's formula:

$$e^{inx/L} = \cos \left( \frac{inx}{L} \right) + i \sin \left( \frac{inx}{L} \right)$$

Two important properties of the set of functions $\{ e^{inx/L} \}$:

1. They are "complete": Any periodic, square integrable function can be written as a sum of the $\{ e^{inx/L} \}$

2. They are "orthonormal": If we multiply two of these functions together and integrate, we get:

$$\int_{-L}^{L} \left( e^{inx/L} \right) \left( e^{-inx/L} \right) \, dx = \begin{cases} 0, & \text{if } m \neq n \\ 2L, & \text{if } m = n \end{cases}$$

We write this:

Define "Kronecker Delta" $\delta_{mn}$:

$$\delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

Then we can write

$$\frac{1}{2L} \int_{-L}^{L} \left( e^{inx/L} \right) \left( e^{-inx/L} \right) \, dx = \delta_{mn}$$

Orthonormality condition for the functions $\{ e^{inx/L} \}$.
Question: How do we find the correct \( \{c_n\} \) to represent a particular function \( f(x) \)?

Answer: Use "Fourier's Trick" method in Griffith's terminology.

Evaluate this integral:

\[
\frac{1}{2L} \int_{-L}^{L} f(x) e^{-\text{i} \pi n x / L} \, dx = \frac{1}{2L} \int_{-L}^{L} \left( \sum_{n=-\infty}^{\infty} c_n e^{-\text{i} \pi n x / L} \right) e^{\text{i} \pi n x / L} \, dx
\]

Fourier series for \( f(x) \):

\[
= \sum_{n=-\infty}^{\infty} c_n \frac{1}{2L} \int_{-L}^{L} \left( e^{-\text{i} \pi n x / L} \right) e^{\text{i} \pi n x / L} \, dx
\]

Kronecker Delta!

\[
= \sum_{n=-\infty}^{\infty} c_n \delta_{nm} = c_m
\]

Kronecker Delta kills all terms except \( n = m \)

Conclusion: To calculate a particular coefficient \( c_m \), do this:

\[
c_m = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-\text{i} \pi m x / L} \, dx
\]

One particular member of the set \( \{c_n\} \).
Example

\[ f(x) = \begin{cases} 0, & -L < x < -\frac{L}{2} \\ 1, & -\frac{L}{2} < x < \frac{L}{2} \\ 0, & \frac{L}{2} < x < L \end{cases} \]

Then

\[ C_m = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-im\pi x/L} \, dx \]

\[ = \frac{1}{2L} \left[ \frac{1}{i m \pi} \left( e^{-im\pi x/L} \right) \right]_{-L}^{L} \]

\[ = \frac{-1}{2i m \pi} \left( e^{-im\pi/2} - e^{im\pi/2} \right) \]

\[ C_m = \frac{1}{m \pi} \sin \left( \frac{m \pi}{2} \right) \quad \text{true for } m \neq 0. \]

We can convert to the sine & cosine form:

For \( m = 0 \): \[ C_0 = \frac{1}{2L} \int_{-L}^{L} f(x) e^{0} \, dx = \frac{1}{2} \]

Convert to sine & cosine form:

\[ a_n = C_n - C_{-n} = \frac{1}{m \pi} \sin \left( \frac{m \pi}{2} \right) + \frac{1}{m \pi} \sin \left( -\frac{m \pi}{2} \right) = \frac{2}{m \pi} \sin \left( \frac{m \pi}{2} \right) \]

\[ b_n = i \left( C_n - C_{-n} \right) = 0 \quad \text{no sine terms!} \]

\( f(x) \) is even, but sine terms are odd.
Summary: For the square wave,
\[ f(x) = \sum_{n=1}^{\infty} \left( \frac{1}{n\pi} \sin\left(\frac{n\pi}{L}\right) \right) e^{in\pi x/L} + \frac{1}{2} \]
except \( n = 0 \) term

AND ALSO
\[ f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \left( \frac{2}{n\pi} \sin\left(\frac{n\pi}{L}\right) \right) \cos\left(\frac{n\pi x}{L}\right) \]
Recap: Let $f(x)$ be (i) periodic with period $2L$ and (ii) square integrable between $-L$ and $L$.

Then $f(x)$ can be written as a "Fourier Series":

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\pi nx/L}$$

for some set of coefficients $\{c_n\}$.

This is very useful for periodic functions. But what if we want to represent a non-periodic function?

In that case we can use the continuous limit of the Fourier series, called the "Fourier Transform".

Rewrite the Fourier Series this way:

$$f(x) = \frac{1}{L} \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} c_n e^{i\pi nx/L} \Delta n = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} c_n e^{i\pi nx/L} \left(\frac{\pi \Delta n}{L}\right)$$

Define:

$$k = \frac{n\pi}{L}, \quad \Delta k = \frac{\pi \Delta n}{L}, \quad \text{and} \quad A(k) = \frac{1}{\pi} c_n.$$

Then,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(k) e^{ikx}\Delta k$$

Now, to represent a non-periodic function, take the limit where the period goes to infinity, $L \to \infty$:
Then \( \Delta k = \frac{2\pi \Delta n}{L} \rightarrow dk \) as \( L \to \infty \)

And

\[
\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk A(k) e^{ikx}
\]

\[\leftarrow\text{Fourier representation of non-periodic } f(x).\]

The function \( A(k) \) is called the 
\textbf{Fourier Transform of } \( f(x) \).

\( A(k) \) is analogous to the coefficients \( \{c_n\} \), but the \( \{c_n\} \) are for representing a periodic function, and \( A(k) \) is for a non-periodic function.

How do we find the correct \( A(k) \) for a particular function \( f(x) \)?

\underline{Answer}: For the \( c_n \), we did this:

\[
c_n = \frac{1}{L} \int_{-L}^{L} f(x) e^{-inx/L} \, dx
\]

Take the limit where \( L \to \infty \):

\[
\hat{c}_n (\frac{L}{\pi}) = \frac{\sqrt{2\pi}}{2\pi} \int_{-L}^{L} f(x) e^{-ikx} \, dx
\]

\[\leftarrow \text{A}(k)\]

\[
A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx
\]

Take the limit where \( L \to \infty \):

\[
A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx
\]

\[\leftarrow \text{Plancherel's Theorem}\]
Example: Let \( f(x) = e^{-ax^2} \) (a gaussian)

What is the Fourier Transform \( \mathcal{F}(k) \)?

Answer:

\[
\mathcal{F}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2-ikx} \, dx
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2+ikx} (-ik) \, dx
\]

Complete the square. Define \( y = \sqrt{a} (x + \frac{ik}{2a}) \).

Then \( y^2 = ax^2 - \frac{k^2}{4a} + ikx \), \( dy = \sqrt{a} \, dx \)

\[
\mathcal{F}(k) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{a}} e^{-\frac{k^2}{4a}} \int_{-\infty}^{\infty} e^{-ax^2+ikx} (-ik) \, dx
\]

\[
= \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi}a} \int_{-\infty}^{\infty} e^{-y^2} \, dy
\]

\[
\mathcal{F}(k) = \frac{e^{-\frac{k^2}{4a}}}{\sqrt{2\pi}a}
\]

The Fourier Transform of a gaussian in \( x \) is a gaussian in \( k \).