The first basic topic will be on mechanics with an emphasis on approximate methods.

Why approximate methods?
- In real life few solvable problems (if any!)
- Even if a problem is solvable approximate solutions may be simpler and help provide insight.

Distinguish two kinds of approximation
- Numerical (as on a computer) errors necessarily exist.
  E.g., how does a computer integrate?

\[
\int_a^b f(x) \, dx \approx \sum \frac{f(x_i)\Delta x}{n}
\]

(Actually there are more sophisticated ways.)

Note we can get more accuracy by making small intervals but it costs computer time.
in principle we can get as much accuracy as we need, but in practice for complicated problems we are limited to analytic - systematic approximations are often based on expansions. Paradigm is Taylor series.

Basic plan: suppose the problem you want to solve is "close" to a problem you can solve.

\[ \text{Problem} = \text{Problem}_0 + \lambda \text{Problem}_1 \]

(Symbolic)

where you know how to solve problem and \( \lambda \) is small, then answer is given by

\[ \text{Answer} = \text{Answer}_0 + \lambda \text{Answer}_1 + \lambda^2 \text{Answer}_2 + \cdots \]

where Answer can be solved with knowledge of Answer, Answer solved with knowledge of Answer and Answer, etc.
generically this is called a perturbation theory or a perturbative expansion

before applying this to real problems let's play with a toy problem

suppose we know that a physical system behaves as follows

\[ V = V_0 e^{-t/\tau} \quad \text{eg.} \quad \frac{dV}{dt} = -\frac{t}{\tau} V \]

where \( V \) is a velocity, \( V_0 \) is an initial velocity and \( \tau \) a parameter with dimension of time and is the "characteristic" time in problem

what does an approximate solution valid for short times look like?

first step — introduce dimensionless variables (always a good practice)

- physics can always be expressed as dimensionless ratios
- simplifies analysis

\[ \tilde{V} = \frac{V}{V_0} \quad \tilde{t} = t/\tau \]

\[ V = e^{-\tilde{t}} \]
what do I mean by "small" times?

\[ \varepsilon \ll 1 \ (i.e., \ v \ll \Omega) \]

now what is my approximation

\[ \hat{v} = \left. \frac{\varepsilon \Omega}{v} \right|_{v = 0} \]

if \( \varepsilon \Omega \rightarrow \infty \) I get exact result

why stop it finite \( \varepsilon \Omega \) ? sloth!

Now key question: how many terms do I need to get a good approximation?

Answer: it depends on what you mean by "good"

see graph

- note for small \( \varepsilon \) works well with few terms as \( \varepsilon \) increases need more terms

as \[ \varepsilon \gg 1 \] need many terms to get decent description thus scheme only really useful for small \( \varepsilon \) in principle if I work...
\textbf{In[20]}: \texttt{Plot}\{\{v, v0, v1, v2, v3, v4, v5, v6, v7, v8\}, \{t, 0, 3\}\}

\textbf{Out[20]}: \texttt{Graphics}
hard and include enough terms I can get good description for larger $\varepsilon$. Why?

$-e^{-\varepsilon}$ converges for any $\varepsilon$

Suppose I had a different problem, suppose

\[ \nu = \frac{\nu_0}{1 + \nu_0 \varepsilon} \]

\[ \frac{d\nu}{dt} = \frac{-\nu_0 / \nu}{(1 + \nu_0 \varepsilon)^2} \]

- dimensionless variable \( \tilde{\nu} = \frac{\nu_0}{\varepsilon} \)

\[ \tilde{\nu} = \frac{1}{1 + \tilde{\varepsilon}} \]

\[ \tilde{\nu} = \sum_{i=0}^{\infty} (-\tilde{\varepsilon})^i \quad \text{(approximate)} \]

see graph

Note that no matter how many terms we include it won't help for $\tilde{\varepsilon} > 1$ ($\varepsilon > \tau$). Why not? For $\tilde{\varepsilon} > 1$ series diverges, i.e., radius of convergence seems to be one.
In[41]:= Plot[{v0, v1, v2, v4, v8, v16, v32, v64, v128}, {t, 0, 3}, PlotRange -> {-3, 3}]
Is idea of expansion as series useless for this problem for $\tilde{\varepsilon} \geq 1$?

No! But need to expand differently — expand in $1/\tilde{\varepsilon}$

\[
\tilde{V} = \frac{1}{1 + \varepsilon} = \frac{1}{\varepsilon (1 + \frac{1}{\varepsilon})} = \frac{1}{\varepsilon} \cdot \frac{1}{1 + \frac{1}{\varepsilon}}
\]

Series

\[
\tilde{V} = -\frac{\tilde{\varepsilon}}{2} (\frac{1}{\varepsilon})^{i+1}
\]

converges just fine for $\tilde{\varepsilon} \gg 1$ ($\varepsilon > 1$)

works well with a few terms for $\tilde{\varepsilon} \gg 1$ ($\varepsilon \gg 1$)

Lessons:

- extract characteristic scales from problem and work with dimensionless variables (keeps you sane)

- do an expansion appropriate for regime in which you are working (eg. $\varepsilon$ for small $\tilde{\varepsilon}$; $1/\varepsilon$ for large $\tilde{\varepsilon}$)
Aside — Sometimes divergent series are useful if truncated so-called asymptotic series.

First a terms get closer to true function then begins to diverge "asymptotic series."

How can it happen?

\[ P(x) = \sum_{j=0}^{\infty} c_j x^j \]

Suppose \( c_j \sim j! \) (faster than power but the series looks like it is convergent for small \( j_{\text{max}} \) but eventually \( c_j^2 \) wins and series diverges.

Example

\[ P(x) = \int_{-\infty}^{\infty} e^{-(x^2 + 2x^4)} \, dx \]
first Taylor expand integrand and then integrate

\[ f(g) = \sqrt{\pi} \left( 1 - \frac{3}{4} g + \frac{105}{32} g^2 - \frac{2435}{128} g^3 + \frac{675675}{2048} g^4 + \ldots \right) \]

coefficients grow so rapidly that series diverges for any \( g \) but look at plot:

for small \( g \) low order expansion describes functions well

seems weird but happens all the time in physics (e.g. perturbative quantum electrodynamics)

why was radius of convergence for

\[ f(g) = \int_0^\infty e^{-x^4 + 2x^2} \frac{dx}{2} \]

finite for any positive or zero \( g \) but \( f(g) \) infinite for any negative \( g \)?

radius of convergence is distance around zero that expansion makes sense but that is zero since for any neg \( g \) we get divergence.
Apply these ideas to a physics problem:

extended object falling through a fluid (e.g., air) subject to resistance due to the fluid

general comments —
- not a full microscopic description
don't need to describe every molecule in fluid
- macroscopic, phenomenological description

\[ \vec{F} = -\alpha \vec{V} \]

- force of resistance proportional to velocity
- coefficient \( \alpha \) depends on shape and size of object, density of material, mass of molecules in fluid, temp...

\( \vec{F}_r \) formula is not exact (valid only for small enough \( \vec{V} \)), and less enough times)
why does $F_r$ depend on $\hat{v}$

- molecules moving randomly (heat) and hitting object

- object moves down -- more fluid molecules per time will hit bottom side than top and typical momentum transfers will be larger

- actually the object will bounce around as molecules hit but we get concept of smooth if viewed for times long compared to typical times between collisions

- proportionality to $\hat{v}$ valid for small $\hat{v}$ but more generally it can be a function of $\hat{v}$ (of which the proportionality is just the lowest term is a Taylor series)
Equation of motion — 1 dimensional

\( F = m \cdot g \)

\( (m \cdot g - \alpha \cdot V) = m \cdot \frac{dV}{dt} \)

\( \frac{dV}{dt} = +g - \left( \frac{\alpha}{m} \right) V \)

Some general comments

— in fact, it is easy to solve this exactly we'll pretend for now we don't notice this to practice our approximation ideas

— 1st step will be to identify relevant scales in problem

Time scale: \( \tau = \frac{a}{g} \) dimensional analysis only time in prob

Velocity scale: \( v_c = g \tau = \frac{a}{2} \)

(call it \( v_c \) with niceties of forethought)
study in approximation valid for small $v$, small compared to what?

$v \ll v_e$

implies only valid for small enough times ($v$ will eventually grow) what does small times mean?

t \ll T

Physically – in this regime $F_{\text{force}}$

Force of gravity $\gg$ force of air resistance

reexpress problem using $\frac{v}{v_e}$ as variable

$\frac{dv}{dt} = (g - \frac{g}{m} v)$

$\frac{i}{v_e} \frac{dv}{dt} = \frac{i}{v_e} (g - \frac{g}{m} v)$

$\frac{d(\frac{v}{v_e})}{dt} = \frac{g}{m} (1 - \frac{v}{v_e}) = \frac{i}{T} \left(1 - \frac{v}{v_e}\right)$
Now let's do a perturbation expansion since for \( v \ll v_c \) the force is dominantly due to gravity. Treat force due to air resistance as a perturbation.

Formal trick — multiply air resistance term by \( a \) to count powers of smallness. Then set \( a \) to end of problem.

\[
\frac{d \left( \frac{v}{v_c} \right)}{dt} = \frac{1}{a} \left( 1 - \frac{v}{v_c} \right)
\]

The next step — solve problem for \( d = 0 \) initial velocity.

\[
\left( \frac{v}{v_c} \right) = \frac{v_0}{v_c} + \frac{gt}{v_c}
\]

What does correspond.

\[ v = v_0 + gt \text{ (ouch!)} \]

Simple free fall. We know this.

Now express answer as a Taylor series in \( a \) systematically including resistance.
where \( \frac{\nu}{\nu_c} = (\frac{\nu}{\nu_c})_0 + \lambda (\frac{\nu}{\nu_c})_1 + \lambda^2 (\frac{\nu}{\nu_c})_2 + \cdots \)

where \( (\frac{\nu}{\nu_c})_0 \) solves \( \lambda = 0 \) problem \( (\text{see 11}) \)

\[
\frac{d(\frac{\nu}{\nu_c})}{dt} = \frac{1}{\tau} \left( 1 - \lambda (\frac{\nu}{\nu_c}) \right)
\]

\[
\frac{d}{dt} \left( (\frac{\nu}{\nu_c})_0 + \lambda (\frac{\nu}{\nu_c})_1 + \lambda^2 (\frac{\nu}{\nu_c})_2 + \cdots \right)
\]

\[
= \frac{1}{\tau} - \frac{\lambda}{\tau} \left( (\frac{\nu}{\nu_c})_0 + \lambda (\frac{\nu}{\nu_c})_1 + \lambda^2 (\frac{\nu}{\nu_c})_2 + \cdots \right)
\]

If equation holds for all \( \lambda \neq 0 \) each must hold separately.

For all powers of \( \lambda - \) collect like powers.

\( \lambda^0 \) : \( \frac{d}{dt} (\frac{\nu}{\nu_c})_0 = \frac{1}{\tau} \)

\( \lambda^1 \) : \( \frac{d}{dt} (\frac{\nu}{\nu_c})_1 = \frac{-1}{\tau} (\frac{\nu}{\nu_c})_0 \)

\( \lambda^2 \) : \( \frac{d}{dt} (\frac{\nu}{\nu_c})_2 = \frac{-1}{\tau} (\frac{\nu}{\nu_c})_1 \)

Or more generally for \( n > 0 \)

\[
\frac{d}{dt} (\frac{\nu}{\nu_c})_n = \frac{-1}{\tau} (\frac{\nu}{\nu_c})_{n-1}
\]
\[
\left( \frac{\nu}{\nu_c} \right)_0 = \left( \frac{\nu_i}{\nu_c} \right) + \frac{t}{T} \quad \text{free fall case}
\]

\[
\frac{d}{dt} \left( \frac{\nu}{\nu_c} \right) = \frac{1}{T} \left( \frac{\nu_i}{\nu_c} + \frac{t}{T} \right)
\]

or

\[
\left( \frac{\nu}{\nu_c} \right)_1 = \left( \frac{\nu_i}{\nu_c} \left( \frac{t}{T} \right) + \frac{1}{2} \left( \frac{t}{T} \right)^2 \right)
\]

so good approx for small times

\[
\frac{\nu}{\nu_c} \approx \left( \frac{\nu_i}{\nu_c} + \frac{t}{T} \right) \approx \left( \frac{\nu_i}{\nu_c} \frac{t}{T} + \frac{1}{2} \left( \frac{t}{T} \right)^2 \right) + \ldots
\]

in terms of original variables

\[
\nu = (\nu_i + 2t) - \left( \nu_i \frac{2t}{T} + \frac{3}{2} \left( \frac{t}{T} \right)^2 \right)
\]

free fall \quad \text{1st correction}

- comments - as advertised works for \( \frac{\nu_i}{\nu_c} \ll 1 \), \( \frac{t}{T} \ll 1 \)

otherwise 2nd term is by as 1st
suppose we need more accuracy (or wish to
push to larger times)

\[ \frac{d \left( \frac{v}{v_e} \right)}{dt} = \frac{-i}{\gamma} \left[ \left( \frac{v}{v_e} \right)^2 + \frac{i}{2} \left( \frac{v}{v_e} \right)^3 \right] \]

\[ \left( \frac{v}{v_e} \right)^2 = \frac{1}{2} \left( \frac{v}{v_e} \right)^2 + \frac{i}{\gamma} \left( \frac{v}{v_e} \right)^3 \]

or

\[ v = \left( v_i + gt \right) - \left( \frac{v_i}{m} t + \frac{1}{2} \left( \frac{m}{2} \right)^2 t^2 \right) + \left( \frac{1}{2} \left( \frac{m}{2} \right)^2 + \frac{1}{4} \left( \frac{m}{2} \right)^3 \right) \]

see 1st correction 2nd correction

actually we can do general term

\[ \left( \frac{v}{v_e} \right)_n = \left( \frac{1}{\gamma} \frac{v}{v_e} \left( \frac{t}{\tau} \right)^n + \frac{1}{(n+1)!} \left( \frac{t}{\tau} \right)^{n+1} \right) (-1)^n \]

satisfies

\[ \frac{d \left( \frac{v}{v_e} \right)_n}{dt} = \frac{-i}{\gamma} \left( \frac{v}{v_e} \right)_{n-1} \] and matches down to \( n=1 \)
Thus we can get as much accuracy as we want (in most problems the general solution for the $n^{th}$ term is hard or impossible to find).

**Full answer:**

\[
\left( \frac{v}{v_e} \right) = \sum_{n=1}^{\infty} \frac{v_i}{v_e} \left( \frac{t}{v_e} \right) (-1)^n + \frac{1}{(n+1)!} \left( \frac{t}{v_e} \right)^{n+1} (-1)^n
\]

we can sum this series

\[
= \frac{v_i}{v_e} e^{-\frac{t}{v_e}} + \left(1 - e^{-\frac{t}{v_e}}\right)
\]

(we use fact $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$)

Thus we have an exact solution by summing up approximations!

What happens is $t \to 0$? $\frac{v}{v_e} \to \frac{v_i}{v_e}$

What happens as $t \to \infty$? $\frac{v}{v_e} \to 1$ or $v \to v_e$.

Thus $v_e$ is the "terminal velocity".

Why is a terminal velocity reached?
This notebook plots the approximate and exact expressions for the falling body with air resistance. The initial condition is $v=0$ at $t=0$. The y-axis is $v/vt$ and the x-axis is $t/\tau$.

```
vovt = 1 - e^{-t}
approx[n_] = Sum[-(-t)^m/Factorial[m], {m, 1, n}]
Plot[{vovt, approx[1], approx[2], approx[3], approx[4],
    approx[5], approx[6], approx[7]}, {t, 0, 6}, PlotRange -> {-1, 2}]
```

```
Out[1]= 1 - e^{-t}
Out[2]= 1 - e^{-t} \frac{(1+n) \text{Gamma}[1+n, -t]}{\text{Gamma}[2+n]}
```

```
Out[3]= -Graphics-
```
System accelerates until \( F_g = -F_r \) and total force is zero. Then a constant "terminal" velocity is reached.

\[ m \ddot{v} = -F_r \]

\[ v_t = \frac{m g}{F_r} \]

exact solution found by resumming series. More direct way once idea of terminal velocity is known

\[ \frac{d}{dt} \left( \frac{v}{v_t} \right) = \frac{1}{v_t} \left( 1 - \frac{v}{v_t} \right) \]

trick well \( \delta = \frac{v}{v_t} - 1 \) (fractional difference of velocity to one)

\[ \frac{d\delta}{dt} = \frac{1}{v_t} \delta \]

\[ y = \frac{v_0}{v_t} e^{-\frac{t}{v_t}} \]

approaches terminal velocity exponential
Now let's generalize things:

we took \[ F_r = -\alpha v \]
but this was only a first approx
in general

\[ F_r = F_r(v) = -\alpha v - \alpha_2 v^2 - \alpha_3 v^3 + \cdots \]
\[ \alpha_2 = \frac{\partial F_r}{\partial v} \bigg|_{v=0} \]
what happens if we include whole function

- in general you have to solve the
differential equation
- we can get insight into certain regimes
  based on physical + math reasoning
  regime of interest \( t \to \infty \) (long time behavior)

physically still expect acceleration until a
terminal velocity is reached

\[ mg = -F_r(v) \]

in general this is a nonlinear equation
which requires a numerical solution
various methods (e.g. Newton's method) built into Mathematica.

Suppose $F_r(v) = -\alpha v e^{-v/v_0}$ with $\alpha, v_0$ constant.

$$F_g = -F_r(v) \Rightarrow +mg = \alpha v e^{v/v_0}$$

Need to solve for $v_r$ numerically from this equation.

Define $\left( \frac{v}{mg} \right) = \frac{v_r}{v_r^{(0)}}$ as terminal velocity for linear problem, a known constant.

So

$$1 = \frac{v_r}{v_r^{(0)}} e^{v_r/v_0}$$

Now suppose $v_0 = 5v_r^{(0)}$ we need to solve

$$1 = \frac{v_r}{v_r^{(0)}} e^{v_r/v_0^{(0)}}$$

- Graphical solution [plot RHS and see where it crosses $v_r^{(0)} = v_r$]

- Find Root $\left[ 1 = v_r e^{v_r/5}, \frac{\delta v_r}{0, 25} \right]$

$\Rightarrow v_r > 0.89458$ so $v_r = 0.89458 v_r^{(0)}$
In general, we need numerics but if

\[ u_t \approx u_t^{(0)} \]

perturbative treatment

\[ u_t = u_t^{(0)} + \lambda (u_t - u_t^{(0)}) \]

happens if nonlinear terms are small

\[ F_r = u_t F_r^{(0)} + \frac{1}{2} u_t^2 F_r^{(0)} + \frac{1}{3!} u_t^3 F_r^{(0)} + \ldots \]

\( \lambda \) counts nonlinearities

plug \( \lambda \) expansion into equation for \( u_t \)

\[ \lambda g = - F_r (u_t) \]

\[ \lambda g = -(u_t F_r^{(1)} + \frac{1}{2} u_t^2 F_r^{(2)} + \ldots) \]

\[ \lambda g = -(u_t^{(0)} F_r^{(1)} + \frac{1}{2} (u_t^{(0)} - u_t^{(0)})^2 F_r^{(2)}) + \ldots \]

\[ = -u_t^{(0)} F_r^{(1)} + \lambda [(u_t - u_t^{(0)}) F_r^{(0)} + \frac{1}{2} u_t^{(0)} - u_t^{(0)} F_r^{(0)}] + \ldots \]
\[ \nu: \quad \nu^{(0)} = -\frac{m g}{F_{r}^{(0)}} \]

\[ A_{i}: \quad [\nu_{r} - \nu^{(0)}] F_{r}^{(0)} = -\frac{i F_{r}^{(0)}}{2} \nu_{r}^{(0)} \]

\[ S_{0}: \quad \frac{\nu_{T} - \nu^{(0)}}{\nu_{r}^{(0)}} = -\frac{i F_{r}^{(0)}}{F_{r}^{(0)}} \nu_{r}^{(0)} \]

Consider our case:

\[ F_{r} = \alpha v e^{v v_{0}} = v \left(1 + \frac{v}{v_{0}} + \frac{1}{2} \frac{v^{2}}{v_{0}^{2}}\right) = \psi v + \frac{v^{2}}{2 v_{0}^{2}} \]

So

\[ \nu_{T}^{(0)} = -\frac{\nu v}{\alpha} \quad \nu_{r}^{(0)} = \frac{v_{T}^{(0)}}{v_{0}} \]

In our case, \( \nu_{0} = 5 \nu_{T}^{(0)} \)

\[ \nu_{r} - \nu_{r}^{(0)} = -\frac{1}{5} \nu_{r}^{(0)} \]

\[ v = v_{T}^{(0)} - \frac{1}{5} v_{T}^{(0)} = \frac{1}{5} \nu_{T}^{(0)} + \Theta(x^{2}) \]

Numerically, \( v = 0.8455 \nu_{T}^{(0)} \)

More accurate — work to higher orders.
Summary

- \( v_T \) reached in more general case
- for general situation, \( v_T \) must be found numerically
- for \( v_T \approx v \), perturbative treatment with non-linearities treated as small

How is \( v_T \) reached —

- in general, for all times need to solve full diff eq.
- but for long times, there is a simple treatment (by long times I mean times for which \( \frac{v - v_T}{v_T} \ll 1 \)) near \( v_T \)

\[
F_r(v) = F_r(v_T) + F_r'(v_T)(v - v_T) + \mathcal{O}(v - v_T^2)
\]

now let us look at equation of motion
\[ m \frac{d \nu}{d t} = m g + F_r(\nu) \]

\[ = m g + F_r(\nu_T) + \int F_r'(\nu) (\nu - \nu_T) + \int \theta (\nu - \nu_T)^2 \]

by definition

\[ = 0 \]

neglected when \( \nu \) close enough to \( \nu_T \)

\[ \frac{d (\nu - \nu_T)}{d t} = \frac{1}{m} F_r'(\nu_T) (\nu - \nu_T) + \text{neglible} \]

\[ (\nu - \nu_T) = (\nu_T - \nu_T) e^{\frac{1}{m} F_r'(\nu_T) t} \]

\[ \nu = (\nu_T - \nu_T) e^{\frac{1}{m} F_r'(\nu_T) t} + \nu_T \]

the system approaches terminal velocity exponentially

\[ \text{eg.} \quad F_r = \text{cubic} \quad \nu_T = \nu_T \]

\[ \text{large nonlinearity} \]

\[ \text{numerically} \]
\[
\frac{dv}{dt} = mg + F_r(v) \\
= mg + \overbrace{F_r(v_\infty) + F_r'(v_\infty)(v - v_\infty)}^{\text{neglect when } v \text{ close enough to } v_\infty} + \alpha(v - v_\infty)^2 \\
= 0 \\
\text{by definition}
\]

\[
\frac{d(v - v_\infty)}{dt} = F_r'(v_\infty)(v - v_\infty) + \text{negligible}
\]

\[
(v - v_\infty) = (v_\infty - v_\infty) e^{F_r'(v_\infty)t} +
\]

\[
v = (v_\infty - v_\infty) e^{F_r'(v_\infty)t} + v_\infty
\]

the system approaches terminal velocity exponentially.

\[
\text{EJ: } F_r = \text{avg} \text{ of } F_r(v) \text{ at } v_\infty \text{ for large non-Linearity}
\]

\[
\text{Example: } F_r(v) = k(v - v_\infty) + \alpha(v^2 - v_\infty^2)
\]

\[
\text{numerically}
\]

\[
\text{subject to}
\]

\[
\beta = \frac{v}{v_\infty}
\]
Approximate solution for system near terminal velocity

• Find terminal velocity for Fr = a \times v^2 \text{ in units of } vr = \frac{av}{mg}

\text{In[1]:=} \text{FindRoot}[1 = vr e^\nu, \{vr, 0, 1\}]
\text{Out[1]:=} \{vr \to 0.567143\}

• Find the derivative of \( F \) at terminal velocity

\text{In[2]:=} \text{dp} = \text{D}[\text{-}vr e^\nu, vr] /. \{vr \to 0.5671432903359215\}
\text{Out[2]:=} 2.95022

• Approximate Solution

\text{In[3]:=} \text{vapp} = (v_i - vr) e^{pt} + vr /. \{vr \to 0.5671432903359215\}
\text{Out[3]:=} 0.557163 + e^{2.95022 t} (-0.557143 + v_i)

• Numerical Solution for \( v_i=0.5 \)

\text{In[50]:=} \text{sol1} = \text{NDSolve}[\{v[x][t] == 1 - vr[t] e^\nu[t], vr[0] == 0.5, vr, \{t, 0, 2\}\}]
\text{Out[50]:=} \text{Plot1} = \text{Plot}[\text{Evaluate}[v[x][t] /. \text{sol1}], \{t, 0, 1.999\}, \text{PlotRange} \to \{0, 1\}]
\text{Plot2} = \text{Plot}[vapp /. \{v_i \to 0.5\}, \{t, 0, 2\}, \text{PlotRange} \to \{0, 1\}]
\text{Show[Plot1, Plot2]}
solutions on top of each other!!

- Numerical Solution for \( vi = 0.47 \)

\[
\text{sol1} = \text{NDSolve}\left[ \{ \text{vr}'[t] == 1 - \text{vr}[t] \text{ e}^{\text{vr}[t]}, \text{vr}[0] == 0.47 \}, \text{vr}, \{t, 0, 2\}\right]
\]

\[
\text{In}[56]:=\quad \text{Plot1} = \text{Plot}\left[ \text{Evaluate}[\text{vr}[t] /. \text{sol1}], \{t, 0, 1.999\}, \text{PlotRange} \to \{0, 1\}\right]
\]

\[
\text{Plot2} = \text{Plot}\left[ \text{vapp} /. \{\text{vi} \to 0.47\}, \{t, 0, 2\}, \text{PlotRange} \to \{0, 1\}\right]
\]

\[
\text{Show}[\text{Plot1}, \text{Plot2}]
\]
New topic:

- Harmonic and anharmonic motion

- Why study harmonic motion?

Consider general 1-d conservative system:

\[ F(x) = -\frac{du}{dx} \]

Why?

Now look at \( u \) near its min.

Call min \( x_0 \)

\[ u = u(x_0) + \frac{1}{2!} u''(x_0) (x-x_0)^2 + \frac{1}{6} u'''(x_0) (x-x_0)^3 + \ldots \]

But by definition, \( u'(x_0) = 0 \) (defines min)

and suppose we are close enough to \( x_0 \) so that \( (x-x_0)^3 \) term can be neglected, then

\[ u = u(x_0) + \frac{1}{2} u''(x-x_0)^2 \]
so \[ F(x) = -\frac{d^2Y}{dx^2} = -\theta Y''(x) \quad (x-x_0) \]

Now let's make a change of variables \( x' = x-x_0 \). 

\[ F(x') = -\theta u''(x_0) \quad x' \]

but this is a harmonic force

\[ m \ddot{x'} = F = -u''(x_0) \quad x' \quad \text{but } u''(x_0) \text{ is a constant, let relabel it } k \]

\[ \ddot{x'} = -\frac{k}{m} x' \]

For simplicity let us drop \( x_0 \) (equivalent to choosing \( x_0 = 0 \) as our origin)

\[ x' = \frac{k}{m} x \]

solutions \( \text{real const. of integration} \)

\[ x = A \cos(\omega t + \delta) \quad (\sqrt{\frac{k}{m}}) \]

or

\[ x = A_c \cos(\omega t) + \theta A_s \sin(\omega t) \]
or
\[ x = r_0 [A \ e^{-\imath \omega t}] \] complex

these three forms are equivalent, why?

get harmonic motion near min of any potential (except pathologigal case of \( V''(r) = 0 \))

why? Any potential looks quadratic

near its min

\[ V(r) \approx \nabla^2 \]

why is region near min interesting? systems tend to settle there if this is any dissipative, in system rocks fall to bottoms of mountains
\[ u = -\frac{u_0}{1 + \left(\frac{x}{l}\right)^2} = -u_0 + \frac{u_0}{l^2} x^2 + -\frac{u_0}{l^4} x^4 \]

Clearly, minimum of potential is at \( x = 0 \)

\[ u''(x_0) = k = \frac{2u_0}{l^2} \quad \text{curvature} \]

\[ x(x) = \frac{A \cos \left( \sqrt{\frac{2mu_0}{A^2}} t + \phi \right)}{\sqrt{A^2}} \]

Suppose \( t = 0 \) \( x(t) = X_0 \) \( \phi \) \( \text{rest} \)

\( t = 0 \) \( x'(t) = 0 \)

\[ X(t) = X_0 \cos \left( \sqrt{\frac{2mu_0}{A^2}} t \right) \]

Valid for small \( X_0 \).
small compared with what?

\[
\frac{x}{l_0} \ll 1
\]

How do I treat the general case?

1) solve differential equation numerically

- basic idea: convert differential equation to difference equation (many ways to do this to minimize errors)

   NDSolve in Mathematica

2) use energy conservation (method of quadratures)

   works for my 1-d conservative problem

   conservative force \( \rightarrow \) energy is conserved

   \[
   E = \text{Kinetic Energy} + \text{Potential}
   \]

   \[
   = \frac{1}{2} m v^2 + U(x)
   \]

   \[
   = \frac{1}{2} m x^2 + U(x)
   \]
\[ \frac{1}{2} m x^2 = E - u \beta \]

\[ x = \sqrt{\frac{2(E - u \beta)}{m}} \]

\[ \frac{dx}{dt} = \pm \sqrt{\frac{2(E - u \beta)}{m}} \quad \text{separate variable} \]

\[ \int \frac{dt}{\sqrt{2(E - u \beta)}} = \int \frac{dm}{\sqrt{2 \left( \frac{m^2 v^2}{c^2} + 4m^2 \beta^2 \right) - v^2 \beta^2}} \]

so

\[ t = \int \frac{x}{\sqrt{2(E - u \beta)}} \, dx = \int \frac{1}{\sqrt{2 \left( \frac{m^2 v^2}{c^2} + 4m^2 \beta^2 \right) - v^2 \beta^2}} \, dx \]

Thus for any given position \( x \), we can determine the time \( t \) at which it gets there.

Initial conditions \( x(0), x(\xi) \rightarrow x(\xi), E \) and fix integrals

\[ \text{issues: we get } t(x) \text{ this way not } \]
\[ x(t) \text{ as is conventional. Need to invert this not always easy analytically, but is trivial numerically} \]
\[ \text{a integral not always doable analytically but generally easy numerically} \]
\[ x(t) \text{ is a function in the sense} \]
that for every $t$ there is one and only $x$. Converse is not true in general: for every $x$ there is not necessarily one $t$.

For confined motion in the potential well motion is periodic. How do integral handle this? Infinite number of times associated with same position at $x$ we get $t = n + \frac{\pi}{\phi}, n = 2, \ldots$.

The issue is associated with sign ambiguity in square root: strictly $\frac{dx}{dt} = \pm \sqrt{\frac{x - x_0}{m}}$

Which do I use, $+$ or $-$?

It depends on the direction. If $t = 0$ and $x = x_0$ (greater than zero) then it is $x = 0$ moving to left and I would use $-$ sign.

I keep using $-$ sign until I reach the turning point (i.e. the point at which $y = \frac{\pi}{2}$) then I use $+$.

Summary — I can solve dif eq. numerically or convert to an integral using energy conservation to solve exact prob.
- these exact methods are numerical or there only approximate methods good
  for small amplitudes - small non-linearities

- concentrate here on symmetric potentials

\[ u(x) = u(-x) \]

\[ u(x) = u_0 + \frac{1}{2} k x^2 + \frac{\alpha}{4!} x^4 + \ldots \]

\[ k = u''(0) \quad \alpha = u^{(4)}(0) \]

I will only work to order of leading correction so I will drop 6th derivative term.

Eg. for \( u = \frac{-u_0}{1 + \beta^2 x^2} = u_0 + \frac{u_0}{\beta^2} x^2 - \frac{u_0}{\beta^4} x^4 \)

So \( k = \frac{2u_0}{\beta^2} \quad \alpha = -\frac{2\beta u_0}{\beta^4} \)

- Hamiltonian formulation

\[ H = \frac{1}{2} \frac{d^2}{dx^2} + \frac{\alpha}{\beta^2} x^2 \]

\[ x = -u_0 x - \frac{\beta x^3}{3 \theta} - \omega_0^4 x^3 \]

\[ \omega_0 = \frac{\beta}{\theta} \]
First attempt at approx naiv perturbation theory

\[ \dot{x}' = -w_0^2 x - \frac{1}{\epsilon} \left( \frac{V}{\epsilon} \right) x^3 \]

Assume \( x = x_0 + Ax_1 + A^2 x_2 + \ldots \)

Work to order \( \lambda \)

\[ \frac{d^2}{dt^2} (x_0 + Ax_1 + \theta \xi) = -w_0^2 (x_0 + Ax_1 + \theta \xi) - \lambda \left( \frac{\lambda}{\epsilon^2} \right) (x_0 + Ax_1 + \theta \xi^2) \]

\[ \frac{d^2 x_0}{dt^2} = -w_0^2 x_0 + \lambda \left( -w_0^2 x_1 + \frac{A^2}{\epsilon^3} x_0 \right) + \theta \xi \]

Equation for \( x \) is ugly, but analytic solution is possible (just ask mathematician using 0 solve but actually we can learn a lot without explicit solution)
form of the equation

\[ x''_i = -\omega_0^2 x_i + \alpha f(t) \]

driven linear oscillator

form of solution

\[ x_i(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t) + \alpha x'_{i,\text{ind}}(t) \]

key point: part of \( x_i \) independent of initial condition, \( \omega_0 \) depends on initial condition

\( \omega_0 \) independent of initial condition (steady state part if damping is present) is proportional to driving term (i.e. \( \alpha \))

why: linear equation = double driving term

\[ x''_{i,\text{ind}} - \omega_0^2 x_{i,\text{ind}} + \alpha f(t) \]

has a cancelling term

so solve for any \( \alpha \) and scale
from form of answer it is clear that approximation breaks down for $t \sim \frac{\pi}{\omega_0} \approx \tau_0$

why suppose $t = 0 \Rightarrow x = 0 \Rightarrow x = x_0$

$0^{th}$ order solution $X(t) = A \times t \cos(\omega_0 t)$

Note satisfies b.c.

$1^{st}$ order solution $X(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t) + x_i^{ind}(t)$

b.c. $X(0) = 0$, $\dot{X}(0) = 0$ so $x_0$ satisfies b.c.

b.c. imply $A = -\alpha x_i^{ind}(0)$

$B = -\frac{\alpha}{\omega_0} \dot{x}_i^{ind}(0)$

or

$x_i(t) = \alpha \left[ -x_i^{ind}(0) - \frac{\dot{x}_i^{ind}(0)}{\omega_0} + x_i^{ind}(t) \right]$

so $X_i(t) \sim \alpha$

Now consider $t = \tau_0 \approx \frac{\pi}{\omega_0}$

$X_i(\tau_0) = \alpha \left[ -x_i^{ind}(0) - \frac{\dot{x}_i^{ind}(0)}{\omega_0} + x_i^{ind}(\tau_0) \right]$

so sign $X_i(\tau_0)$ determined by sign of $\alpha$
\[ X = X_0 + X_1 \]
\[ x(t) = X_0 \cos(\omega_0 t) + X_1 (t) = X_0 \cos(\omega_0 t) + x(t) \]

However, I know by energy conservation for any time

\[ X < X_0 \] by energy conservation

At \( t = x_i \), we have zero kinetic energy but increased \( x \), hence pot energy

\[ V(x_i) \]

So system will always turn around at \( x_i \)

Now, we have a contradiction—

For some \( \omega_0 \) and \( x \), we always get \( X \geq x_i \) since \( x_i (t) \to \infty \)

Naive perturbation theory cannot be valid for \( t \to T \) we conclude it work only for \( t \ll T \)
- This is not very useful!!

- How could it happen?

Consider the following function

\[ x(t) = A \cos((ω₀ + Δω)t) \]
\[ = A \cos(ω₀t) \cos(Δωt) - A \sin(ω₀t) \sin(Δωt) \]

as \( t \to 0 \) simple \cos behavior

do the Taylor series

\[ x(t) = A \cos(ω₀t) \]
\[ + \frac{Δω}{ω₀} A \sin(ω₀t) \]
\[ + \frac{1}{2} \frac{Δω^2}{ω₀^2} A \cos(ω₀t) \]
\[ + \cdots \]

This clearly fails for \( t \to 0 \)

Problem - system is periodic but we are expanding about the wrong period!

Problem is we need to build up \cos(Δωt) as a series
Lesson - sometimes clever screening schemes do not work for regime of interest

How do we get good expansion schemes?
- build in fact that system is periodic
- expand around the right period

Problem: we don't know what right period is - if \( q \neq \pi \)
Solution: expand based on an unknown period

Needed tool - Fourier Analysis

Review:
expand any periodic function in terms of a complete set of functions

\[
\text{if} \quad f(t) = f(t + \tau)
\]

then \[
f(t) = \sum_{n=0}^{\infty} a_n \cos \left( \frac{2\pi nt}{\tau} \right) + \sum_{n=0}^{\infty} b_n \sin \left( \frac{2\pi nt}{\tau} \right)
\]

real form,

\[
\sum_{n=\infty}^{\infty} C_n e^{i \left( \frac{2\pi nt}{\tau} \right)}
\]

complex form.
each term in series is periodic so $\sin t$ is
higher terms in $a_n$ are more rapidly varying
small $n$ - smooth
large $n$ - finer structure

how do we extract $a_n, b_n$ (or $c_n$)

use following facts

$$\frac{1}{\pi} \int_0^\pi \cos \left( \frac{2\pi nt}{\pi} \right) \cos \left( \frac{2\pi n_0 t}{\pi} \right) \, dt = \delta_{nn_0} \text{ for } n_0 \neq 0$$

$$= 2 \text{ for } n = n_0$$

$$= \delta_{nn_0} (1 + \delta_{n0})$$

$$\frac{1}{2\pi} \int_0^\pi \cos \left( \frac{2\pi nt}{\pi} \right) \sin \left( \frac{2\pi n_0 t}{\pi} \right) \, dt = 0$$

$$\frac{1}{2\pi} \int_0^\pi \sin \left( \frac{2\pi nt}{\pi} \right) \sin \left( \frac{2\pi n_0 t}{\pi} \right) \, dt = \delta_{nn_0} (1 - \delta_{n0})$$

$$\left( \frac{1}{\pi} \int e^{-\frac{2\pi nt}{\pi}} \, e^{i\pi \frac{2\pi n_0 t}{\pi}} \, dt \right) = \delta_{nn_0} \delta_{n0}$$

so for real case multiply both sides
by $\frac{1}{2\pi} \cos \left( \frac{2\pi nt}{\pi} \right)$ and integrate from $0$ to $\pi$

$$a_n = \frac{2}{\pi} \int_0^\pi f(t) \cos \left( \frac{2\pi nt}{\pi} \right) \, dt \frac{1}{1 + \delta n_0}$$

$$b_n = \frac{2}{\pi} \int_0^\pi f(t) \sin \left( \frac{2\pi nt}{\pi} \right) \, dt$$
or using complex formulation

\[ c_n = \frac{1}{T} \int_0^T f(t) e^{-i n \omega t} \, dt \]

Analogy — looks like picking out coefficients in a vector space.

In n dimensions

\[ x = \sum_{n} a_n \hat{e}_n \]

if all \( \hat{e} \)'s orthogonal,

then

\[ d_n = \hat{e}_n \cdot x \]

so functions \( \sin \left( \frac{2\pi n t}{T} \right) \), \( \cos \left( \frac{2\pi n t}{T} \right) \)

act like basis vectors

and \( \frac{1}{T} \int_0^T f(t) \, dt \) act like a product
Scheme — write answer as a Taylor series in an unknown period $\tau$

\[ w = \frac{2\pi}{\tau} \]

\[ x(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} \]

Note — $x$ is real so $x^* = x$

\[ x^* = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} \]

thus $c_n^* = c_{-n}$

Moreover $c_0 = 0$ time averaged system has zero displacement as we fixed $x_0 = 0$

\[ x = \sum_{n=1}^{\infty} c_n e^{i\omega_n t} + h.c. \]

Now we know system is harmonic for small amplitudes. So a reasonable series would have n=1 terms lead with n=2 as corrections, etc.

\[ x = \sum_{n=1}^{\infty} c_n \lambda^n e^{i\omega_n t} + h.c. \]
Let's look at equations of motion.

\[ \ddot{x} = -\omega_0^2 x - \frac{k}{m} x^3 \]

Plug in \( \lambda \) and work to order \( \lambda^3 \)

\[ \lambda \left[ c_1 \left( -\omega_0^2 e^{-i\omega t} \right) + h.c. \right] = -\omega_0^4 \left[ c_1 e^{-i\omega t} + h.c. \right] \]
\[ + \lambda^3 c_3 \left( -4 \omega_0^2 e^{-i3\omega t} \right) + h.c. \]
\[ + \lambda^3 c_2 \left( -9 \omega_0^2 e^{-i\omega t} \right) + h.c. \]
\[ + \lambda^3 c_3 \left( -9 \omega_0^2 e^{-i\omega t} \right) + h.c. \]
\[ - \lambda^3 \frac{k}{m} \left[ c_1 e^{-i\omega t} + h.c. \right] \]

Now, \( c_1 e^{-i\omega t} + c_1^* e^{i\omega t} \)

\[ c_0 e^{-i3\omega t} + 3c_1^2 c_1 e^{i\omega t} + 3c_2 c_1 e^{-i\omega t} + c_3 e^{-i\omega t} \]

So now, let \( \omega \to \omega_0 \)

And keep track of Fourier components.

\[ -\omega_0^2 c_1 e^{-i\omega t} = -\omega_0^2 c_1 e^{-i\omega t} + \frac{3k c_2^2 c_1 e^{-i\omega t}}{m} \]
\[ -4\omega_0^2 c_2 e^{-i\omega t} = -\omega_0^2 c_2 e^{-i\omega t} \]
\[ -9\omega_0^2 c_3 e^{-i\omega t} = -\omega_0^2 c_3 e^{-i\omega t} - \frac{k}{m} c_3^2 e^{-i\omega t} \]
\[ + h.c. \]
pick out each fourier component
(if you want $f(t) e^{-i \omega t}$ $\times$ equation

$e^{i \omega t}$ component: $- \omega^2 \xi_1 = - \omega^2 \xi_1 + \frac{3 \alpha}{6 \eta} \xi_1 \xi_3^*$
$e^{-i \omega t}$ component: $- 4 \omega^2 \xi_2 = - \omega^2 \xi_2$
$e^{-i \omega t}$ component: $- \omega^2 \xi_3 = - \omega^2 \xi_3 - \frac{\alpha}{6 \eta} \xi_3^3$

suppose that $\xi_1$ is real (at $t=0$ $x=0$)

1$^{\text{st}}$ equation: $\omega^2 = \omega_0^2 + \frac{\alpha}{2 \eta} \xi_1^2$ \textit{$\leftarrow$} gives shift in

2$^{\text{nd}}$ equation: $\xi_3 = 0$ only odd component \textit{Frequency (or period) from small angle}

3$^{\text{rd}}$ equation: $(9 \omega^2 - \omega_0^2) \xi_3 = - \frac{\alpha}{6 \eta} \xi_3^3$

now $\omega^2 = \omega_0^2 + \frac{\alpha}{2 \eta} \xi_1^2$ ~ neglect for now

$8 \omega_0^2 \xi_3 = - \frac{\alpha}{6 \eta} \xi_3^3$

$\xi_3 = \frac{\xi_1}{8 \eta \omega_0^2 \xi_1^3}$ \textit{$\leftarrow$ 3$^{\text{rd}}$ fourier component in terms of 1$^{\text{st}}$

now we have entire answer in terms of $\xi_1$ we would like if in terms of $x_i$ (recall $x_i = x_1$)
\[ x_i = x(e=0) \]
\[ = 2 c_1 + 2 c_3 \]
\[ c_3 = \frac{\alpha}{48 \pi \omega_0^3} c_1^3 \]

Recall \( c_1 \sim \lambda \), \( c_3 \sim \lambda^3 \), so \( x_i \sim \lambda \)

So
\[ x_i = 2 c_1 + O(\lambda^3) \]
\[ c_1 = \frac{x_i}{2} + O(\lambda^3) \]

or
\[ c_3 = \frac{\alpha x_i^3}{384 \pi \omega_0^3} \left( 1 + O(\lambda^2) \right) \]

So
\[ x_i = \lambda \left( c_1 + c_3 \right) \]
\[ c_1 = \frac{x_i}{\lambda} - c_3 = \frac{x_i}{2} - \frac{\alpha x_i^3}{384 \pi \omega_0^3} + O(\lambda^5) \]

\[ w^{-2} = \omega_0^{-2} \left( 1 + \frac{\alpha}{2 m \omega_0^2} c_1^2 \right) = \omega_0^{-2} \left( 1 + \frac{\alpha}{2 m \omega_0^2} \frac{x_i^2}{4} + O(\lambda^3) \right) \]
\[ = \omega_0^{-2} \left( 1 + \frac{\alpha x_i^2}{8 m \omega_0^2} \right) + O(\lambda^4) \]

\[ w = \omega_0 \left( 1 + \frac{\alpha x_0^2}{8 m \omega_0^2} \right)^{1/2} = \omega_0^{\bullet} + \frac{\alpha x_i^2}{16 m \omega_0^{\bullet}} + O(\lambda^4) \]
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put it all together

\[ x = c_1 e^{i\omega t} + c_1^* e^{-i\omega t} + c_2 e^{i\omega t} + c_2^* e^{-i\omega t} \]

\[ = 2c_1 \cos(\omega t) + 2c_3 \cos(3\omega t) \quad \text{real } c_i \]

\[ = \left( X_i - \frac{\alpha X_i^3}{192m\omega_0^2} \right) \cos \left( \omega_0 + \frac{\alpha X_i^2}{16m\omega_0} \right) t \]

\[ + \frac{\alpha X_i^3}{192m\omega_0^2} \cos \left( 3\left( \omega_0 + \frac{\alpha X_i^2}{16m\omega_0} \right) t \right) \]

summary

- frequency is not \( \omega_0 \) it depends on amplitude
- higher Fourier components added in

Qualitative:

\[ \alpha > 0 \quad \omega > \omega_0 \quad ? \]

\[ \alpha < 0 \quad \omega < \omega_0 \]

why?
Physical Pendulum

\[ I \ddot{\theta} = -\frac{\tau}{I} = -\sigma (\frac{a}{r_c^2} \cdot \frac{a}{r_c^2}) \cdot 2 = -\sigma (mgI \times r_c \cdot \sin \theta) \]

\[ I \ddot{\theta} = -mgI \sin \theta \]

\[ \ddot{\theta} = -\frac{mgI \sin \theta}{I} \]

Now, \[ I = \int dx \cdot r(x) \cdot z^2 \]

considering a massless rod

\[ I = \int dx \cdot \rho(x) \cdot z^2 \]

\[ \frac{\ddot{z} \cdot I}{I} = \frac{\ddot{z} \cdot mgI}{mgI} = \frac{\ddot{z}}{I} \]

\[ \ddot{z} = -\frac{1}{I} mgI \sin \theta \]

small amplitude

\[ \ddot{\theta} \approx -\frac{1}{I} mgI \sin \theta \]

\[ \omega_0 = \sqrt{\frac{g}{l}} \]

general case of pendulum
\[ \theta = -\omega_0^2 \sin \theta \quad \omega_0^2 = \frac{mg}{2I} \]

Small amplitude with correction to rotational one

\[ \ddot{\theta} = -\omega_0^2 (\theta - \frac{1}{6} \theta^3) \]

analogous to

\[ \ddot{x} = -\omega_0^2 x - \frac{1}{6} \frac{\alpha}{m} x^3 \]

The point is that the equation for \( \theta \) is of same form as equation for \( x \) in solution of same form. This isn't even though \( \theta \) and \( x \) don't have same dimensions.

Treat as

\[ \ddot{\theta} = -\omega_0^2 \sin \theta \quad u(\theta) = -\frac{1}{2} \omega_0^2 \cos \theta + \text{const} \]

General problem of

\[ \ddot{x} = -\omega_0^2 \sin \left( \frac{\alpha x}{2} \right) \quad u(x) = -\frac{1}{2} \omega_0^2 \cos \frac{x}{2} + \text{const} \]

\[ x \leftrightarrow \theta \]
Physically, a pendulum localized while sinusoidal potential isn't but as \( x \to x + 2\pi \) although we have mixed behavior is same as if we had a (symmetry) just as in the case of \( \theta \to \theta + 2\pi \).

I'll discuss pendulum but applies equally well to sinusoidal potential in space.

**Qualitative regimes:**

1) Bounded motion

2) Unbounded motion

Qualitative quite different — what controls which regime I'm in
total energy is the key

\[ U(\theta) = \frac{I}{2} \omega^2 \cos \theta + \text{const} \]

pick const so that at \( \theta = 0 \), \( \omega = 0 \)

\[ \frac{I}{2} \omega^2 (1 - \cos \theta) \]

what is max of \( U(\theta) \)?

\[ U_{\text{max}} = \frac{I}{2} \omega^2 \]

why?

Suppose \( E > U_{\text{max}} \)

\[ E = \frac{1}{2} I \dot{\theta}^2 + U \]

so \( \dot{\theta}^2 = \frac{2(E-U)}{I} \)

so if \( E > U_{\text{max}} \) then \( 0 < E - U < E - U_{\text{max}} \).

Thus \( \dot{\theta}^2 > 0 \) always

\[ \text{system never stops} \]

get unbounded motion
conversely if

\[ E < \text{\textit{u}}_{\text{max}} \] then at some point
\[ \dot{\theta}^2 = 0 \] and system stops and turns around (bounded motion)

quantitatively interesting regimes:

bounded motion:

- small amplitude
  \[ \theta_{\text{amp}} \ll \pi \quad \text{\textit{u}} \ll \text{\textit{u}}_{\text{max}} \]
  already analyzed this case

- very large amplitude (nearly unbounded)
  \[ \theta_{\text{amp}} \approx \pi \quad (\text{strictly just less than } \pi) \]
  \[ E \approx \text{\textit{u}}_{\text{max}} \quad (\text{strictly } E \text{ just less than } \text{\textit{u}}_{\text{max}}) \]

unbounded motion:

- very high energy \( E >> \text{\textit{u}}_{\text{max}} \)
  (restoring force almost negligible)
  \[ \theta(t) = \dot{\theta} + \text{small correction} \]
  (why)
- nearly bounded \( E \approx \text{\textit{u}}_{\text{max}} \) (just above)
Focus here on bounded motion for

\[ E < \frac{1}{2} \omega^2 \] (nearly unbounded)

qualitatively expect long periods in this regime

\[ \gamma \to \infty \quad \text{as} \quad E \to \text{Unsa} \]

why? as \( E \to \text{Unsa} + \epsilon \) system never returns so \( \gamma \to \infty \)

try to understand \( \gamma \) is this regime

- key point most of the time spent near \( \theta = \pm \pi \)

(moves quickly everywhere except near endpoint which is near \( \theta = \pm \pi \))

to estimate time estimate time near endpoint

\[ \max_{\theta} u(\theta) \approx \frac{1}{2} \omega^2 (1 - \cos \theta) \]

at \( \theta = \pi \)

\[ \approx \frac{K}{2} \omega^2 [2 - \frac{1}{2} (\theta - \pi)^2] + \ldots \]

so equation of motion near max

\[ T \ddot{\theta} = -g \dot{\theta} \implies \ddot{\theta} = \frac{g}{\omega^2} (\theta - \pi) \]

stable point
or \( \frac{d^2(\theta - \pi)}{dt^2} = w_0^2 (\theta - \pi) \) (valid only near \( \theta = \pi \))

\[ \text{Solution} \]

\[ (\theta - \pi) = A e^{w_0 t} + B e^{-w_0 t} \]

more convenient

\[ = A_c \cos(w_0 t) + A_s \sin(w_0 t) \]

\[ \cos(w_0 t) = \frac{1}{2} (e^{w_0 t} + e^{-w_0 t}) \]
\[ \sin(w_0 t) = \frac{1}{2} (e^{w_0 t} - e^{-w_0 t}) \]

To estimate \( r \) ask how long it takes to go from turning point to some distance by which it is moving "fast" and multiply by four

- at turning point \( v = 0 \)
- suppose at turning point at \( t = 0 \) find \( A_c, A_s \)

\[ \theta_t = \theta_0 \text{ at turning point} \]

\[ (\theta_t - \pi) = A_c \cosh(0) + A_s \sinh(0) = A_c \]
\[ \Theta(t=0) = 0 = A_c \omega \sinh(0) + A_s \cosh(0) = \omega_0 A_s \]

\[ \therefore (\Theta - \pi) = A_c \quad A_s = 0 \]

Next, turning point if turning point is near \( T_0 \):

\[ (\Theta - \pi) = (\Theta_T - \pi) \cosh \omega_0 t \]

Turning point

On physical grounds, for \( \Theta_T > \pi \), system stays near turning point for a long time.

Long compared to what?

Only scale in problem: \( \frac{\omega}{\omega_0} \equiv \tau_0 \)

- For \( t \gg T_0 \) but still near \( \Theta = \pi \)

\[ \cosh(\omega_0 t) = \frac{1}{2} \left( e^{\omega_0 t/\tau_0} + e^{-\omega_0 t/\tau_0} \right) \]

\[ \therefore \frac{1}{2} e^{\omega_0 t/\tau_0} \]

So

\[ (\Theta - \pi) = (\Theta_T - \pi) \frac{i}{2} e^{2\pi t/\tau_0} \]
time it takes to get from $\theta_0$ to $\theta = \omega$

$$\frac{(\theta - \pi)}{2(\theta_0 - \pi)} = e^{w_0 t_0}$$

$$w_0 t = \log \left[ \frac{\theta - \pi}{\theta_0 - \pi} \right]$$

$$t = \frac{1}{w_0} \log \left[ \frac{2(\theta - \pi)}{(\theta_0 - \pi)} \right] = \frac{1}{w_0} \log \left[ \frac{4(\theta_0 - \pi)}{\pi - \theta_0} \right]$$

valid for $t_0 > \frac{\pi}{w_0}$, $(\theta - \pi) \ll \pi$

period = $4 \times t_0 + 2 \times$ time it takes to get from $\theta_0$ to $\omega$

$$T = 4 \log \left[ \frac{2h - \theta}{\pi - \theta_0} \right] + \text{const}$$

$$4 \log \left( \frac{2h - \theta}{\pi - \theta_0} \right) = \text{number}

= \text{something}

= \theta_0 \log (2h - \theta) - \frac{\pi}{T_0} \log (\pi - \theta_0) + \text{const}

as $\theta_0 \rightarrow \pi$,

$$\log (\pi - \theta_0) \rightarrow -\infty$$

$$\gamma = -\frac{1}{2\pi} \log (\pi - \theta_0)$$

period $\rightarrow 0$ logarithmic
Summary

- We have verified common sense notion that \( z \rightarrow 0 \) is \( x \rightarrow x \)

- We have quantified how

\[
T \sim \sqrt{\frac{\pi}{e}} \log (\pi - \delta_t)
\]

valid near \( \delta_t = \pi \)

- Region of \( E > \text{max} \) (Project!!)