1. [10 pts] Show that the displacement current inside a parallel-plate capacitor can be written as:

\[ C \frac{dV_c}{dt} \]

where \( C \) is the capacitance and \( V_c \) is the voltage across the capacitor.

**Solution:**

We can use one of the Maxwell's equations:

\[ \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_s + \mu_0 \varepsilon_0 \frac{\partial \Phi_{E,S}}{\partial t} \]

and rewrite it as

\[ \oint B \cdot d\mathbf{l} = \mu_0 I_s + \mu_0 I_{Disp} \]

We might say that the magnetic field \( B \) at some point outside the capacitor originates due to the effect of a real current \( I_s \) and a displacement current \( I_{Disp} \).

Then we have

\[ I_{Disp} = \varepsilon_0 \frac{\partial \Phi_{E,S}}{\partial t} = \varepsilon_0 \frac{\partial (\mathbf{E} \cdot \mathbf{A})}{\partial t} = \varepsilon_0 \frac{\partial V_c}{\partial t} = \frac{\varepsilon_0 A}{D} \frac{\partial V_c}{\partial t} = C \frac{\partial V_c}{\partial t} \]

Where we have used the formula for capacitance of a parallel plate capacitor of plate separation \( D \) and plate area \( A \) (which is constant over time),

\[ C = \frac{\varepsilon_0 A}{D} \]

And the relationship between electrostatic field and electrostatic potential,

\[ E_x = -\frac{\partial V_c}{\partial x} = -\frac{V_c}{D} \]

where \( x \) is the direction normal to the plates and the potential gradient is uniform between the plates and we have neglected the negative sign since we are dealing with magnitudes only.