K7-27: RLC CIRCUIT - COMPLETE

K7-45: LOW AND HIGH PASS FILTERS
Chapter 36. AC Circuits

Today, a “grid” of AC electrical distribution systems spans the United States and other countries. Any device that plugs into an electric outlet uses an AC circuit. In this chapter, you will learn some of the basic techniques for analyzing AC circuits.

Chapter Goal: To understand and apply basic techniques of AC circuit analysis.
Chapter 36. AC Circuits

Topics:

• AC Sources and Phasors
• Capacitor Circuits
• $RC$ Filter Circuits
• Inductor Circuits
• The Series $RLC$ Circuit

Skip Section 36.6 – Power factor
**AC Sources and Phasors**

**FIGURE 36.1** An oscillating emf can be represented as a graph or as a phasor diagram.

The emf oscillates as $\mathcal{E} = \mathcal{E}_0 \cos \omega t$.

The oscillation period is $T = 1/f = 2\pi/\omega$. 

**Graph:**
- $\mathcal{E}$ vs. time ($t$)
- Peak emf
- $\mathcal{E}_0$
- $-\mathcal{E}_0$
- Time intervals: $T$ and $2T$
AC Sources and Phasors

**FIGURE 36.1** An oscillating emf can be represented as a graph or as a phasor diagram.

The length of the phasor is $E_0$.

The phasor rotates ccw at angular frequency $\omega$.

The *phase angle* is $\omega t$.

The tip of the phasor goes once around the circle in time $T$.

The instantaneous emf value $E_0 \cos \omega t$ is the projection of the phasor onto the horizontal axis.

**FIGURE 36.1** An oscillating emf can be represented as a graph or as a phasor diagram.

(a) $\mathcal{E}$

- Peak emf
- The emf oscillates as $E = E_0 \cos \omega t$.

(b) The oscillation period is $T = 1/f = 2\pi/\omega$. 
In an AC resistor circuit, Ohm’s law applies to both the instantaneous and peak currents and voltages.
The *resistor voltage* $v_R$ is given by

$$v_R = V_R \cos \omega t$$

where $V_R$ is the peak or maximum voltage. The current through the resistor is

$$i_R = \frac{v_R}{R} = \frac{V_R \cos \omega t}{R} = I_R \cos \omega t$$

where $I_R = V_R / R$ is the peak current.
AC Circuits - Resistors

**FIGURE 36.5** Graph and phasor diagram of the resistor current and voltage. The current and voltage are in phase.

(a) \(v_R\) and \(i_R\)

\[v_R = V_R \cos \omega t\]

\[i_R = I_R \cos \omega t\]

The resistor voltage and current oscillate in phase.

(b) Voltage phasor, length \(V_R\)

Current phasor, length \(I_R\)

Instantaneous current and voltage
AC Circuits - Capacitors

The AC current to and from a capacitor leads the capacitor voltage by \( \pi/2 \) rad, or 90°.

\[ V_C = V_c \cos \omega t = \frac{q}{C} \]

\[ \therefore q = C V_c \cos \omega t \]

\[ \frac{dq}{dt} = i = V_c \omega C \left( -\sin \omega t \right) \]

\[ i_c = I_c \cos \left( \omega t + \frac{\pi}{2} \right) \]

The AC current to and from a capacitor leads the capacitor voltage by \( \pi/2 \) rad, or 90°.

Want "Ohm's Law":

\[ I_C = \frac{V_C}{X_C} \Rightarrow X_C = \frac{1}{\omega C} \]
AC Circuits - Capacitors
Capacitive Reactance

The capacitive reactance \( X_C \) is defined as

\[
X_C \equiv \frac{1}{\omega C}
\]

The units of reactance, like those of resistance, are ohms. Reactance relates the peak voltage \( V_C \) and current \( I_C \):

\[
I_C = \frac{V_C}{X_C} \quad \text{or} \quad V_C = I_C X_C
\]

NOTE: Reactance differs from resistance in that it does not relate the instantaneous capacitor voltage and current because they are out of phase. That is, \( v_C \neq i_C X_C \).
ASSESS Using reactance is just like using Ohm’s law, but don’t forget it applies to only the peak current and voltage, not the instantaneous values.

(a) $i_c$ peaks $\frac{1}{4}T$ before $v_c$ peaks. We say that the current *leads* the voltage by 90°.

(b) The current phasor leads the voltage phasor by 90°. These are the instantaneous current and voltage.
The AC current through an inductor lags the inductor voltage by $\pi/2$ rad, or 90°.

The instantaneous inductor voltage is $v_L = L(di_L/dt)$.

$$v_L = V_L \cos \omega t = L \frac{dI_L}{dt} \quad \Rightarrow \quad i_L = \frac{V_L}{L} \int \cos \omega t \, dt = \frac{V_L}{\omega L} \sin \omega t = \frac{V_L}{\omega L} \cos \left(\omega t - \frac{\pi}{2}\right)$$

$$= I_L \cos \left(\omega t - \frac{\pi}{2}\right)$$

Want "Ohm's Law": $I_L = \frac{V_L}{X_L} \Rightarrow \quad X_L = \omega L$

The AC current through an inductor lags the inductor voltage by $\pi/2$ rad, or 90°.
AC Circuits - Inductors
Inductive Reactance

The inductive reactance $X_L$ is defined as

$$X_L \equiv \omega L$$

Reactance relates the peak voltage $V_L$ and current $I_L$:

$$I_L = \frac{V_L}{X_L} \text{ or } V_L = I_L X_L$$

NOTE: Reactance differs from resistance in that it does not relate the instantaneous inductor voltage and current because they are out of phase. That is, $v_L \not= i_L X_L$. 
Inductive Reactance

\[ i_L = I_L \cos(\omega t - \pi/2), \quad v_L = V_L \cos(\omega t) \]

(a) \( v_L \) and \( i_L \)

- \( i_L \) peaks \( \frac{1}{4}T \) after \( v_L \) peaks.
- We say that the current lags the voltage by 90°.

(b) Voltage phasor

- The current phasor lags the voltage phasor by 90°.
RC Filters – The concept (Fourier analysis)

Any waveform (like voltages driving your speaker when you play music) is a sum of many sinusoidal waveforms of different amplitudes and frequencies.

The ac voltage generator depicted below for an RC circuit is idealized as ONE input frequency, but in general could be a sum of MANY waveforms (like music) with many frequencies.

Goal: Analyze the individual voltages across the resistor and capacitor when an input waveform with any frequency \(\omega\) and voltage amplitude \(E_0\) is applied across both.
RC Filters – Analysis

Analyzing an RC circuit

1. Current is the same at all points in circuit at all time. Choose an arbitrary current vector (time).
2. Phase: \( V_R \) in phase with \( I \), \( I \) in capacitor leads \( V_c \)
   Amplitude: For a given \( I \) peak value, we know \( V_R \) and \( V_c \) peak
3. At any instant in time, we have (Kirchoff’s loop law):
   \[
   \vec{V}_R \cdot \vec{X} + \vec{V}_C \cdot \vec{X} = \vec{E} \cdot \vec{X} \Rightarrow \vec{E} = \vec{V}_R + \vec{V}_C
   \]
   \( E_0 \cos \omega t \)
RC Filters – Analysis

Analyzing an RC circuit

\[ \varepsilon_0^2 = V_R^2 + V_C^2 = (IR)^2 + (IX_C)^2 = (R^2 + X_C^2)I^2 \]
\[ = (R^2 + \frac{1}{\omega^2C^2})I^2 \]

Consequently, the peak current in the RC circuit is

\[ I = \frac{\varepsilon_0}{\sqrt{R^2 + X_C^2}} = \frac{\varepsilon_0}{\sqrt{R^2 + \frac{1}{\omega^2C^2}}} \]

Knowing \( I \) gives us the two peak voltages:

\[ V_R = IR = \frac{\varepsilon_0R}{\sqrt{R^2 + X_C^2}} = \frac{\varepsilon_0R}{\sqrt{R^2 + \frac{1}{\omega^2C^2}}} \]
\[ V_C = IX_C = \frac{\varepsilon_0X_C}{\sqrt{R^2 + X_C^2}} = \frac{\varepsilon_0/\omega C}{\sqrt{R^2 + \frac{1}{\omega^2C^2}}} \]
RC Filters – Analysis

\[ V_R = IR = \frac{E_0 R}{\sqrt{R^2 + X_C^2}} = \frac{E_0 R}{\sqrt{R^2 + 1/\omega^2 C^2}} \]

\[ V_C = IX_C = \frac{E_0 X_C}{\sqrt{R^2 + X_C^2}} = \frac{E_0 \omega C}{\sqrt{R^2 + 1/\omega^2 C^2}} \]

For \( \omega \to 0 \) (\( X_C \gg R \)), \( V_R \to 0 \), \( V_C \to \frac{E_0}{X_C} \) – Like Resistor S h o r t e d

For \( \omega \to \infty \) (\( R \gg X_C \)), \( V_R \to \frac{E_0}{R} \), \( V_C \to 0 \) – Like Capacitor S h o r t e d

For \( \omega = \frac{1}{RC} \) (\( R = X_C \)), \( V_R \to \frac{1}{\sqrt{2}} \frac{E_0}{R} \), \( V_C \to \frac{1}{\sqrt{2}} \frac{E_0}{X_C} \)

So \( V_R = V_C \)

\[ \omega_c = \frac{1}{RC} \), C o r s s - o v e r f r e q u e n c y
RC Filters – Analysis

\[ V_R = IR = \frac{E_0 R}{\sqrt{R^2 + X_C^2}} = \frac{E_0 R}{\sqrt{R^2 + 1/\omega^2 C^2}} \]

\[ V_C = IX_C = \frac{E_0 X_C}{\sqrt{R^2 + X_C^2}} = \frac{E_0}{\omega C} \]

For \( \omega \to \infty \) \((X_C \gg R)\): \( V_R \to 0 \), \( V_C \to \frac{E_0}{X_C} \)

For \( \omega \to 0 \) \((R \gg X_C)\): \( V_R \to \frac{E_0}{R} \), \( V_C \to 0 \)

For \( \omega = \frac{1}{RC} \) \((R = X_C)\): \( V_R \to \frac{1}{\sqrt{2}} \frac{E_0}{R} \), \( V_C \to \frac{1}{\sqrt{2}} \frac{E_0}{X_C} \)

So \( V_R = V_C \)

\( W_C = \frac{1}{RC} \), crossover frequency
RC Filters – Analysis

(a) Low-pass filter

\[ v_{\text{out}} = v_C \]

Transmits frequencies \( \omega < \omega_c \) and blocks frequencies \( \omega > \omega_c \).

(b) High-pass filter

\[ v_{\text{out}} = v_R \]

Transmits frequencies \( \omega > \omega_c \) and blocks frequencies \( \omega < \omega_c \).

• Capacitor like a short at high frequencies since:

\[ X_C = \frac{1}{\omega C} \rightarrow 0 \text{ at High } \omega \]

• Voltage across Capacitor dominates at low frequencies since:

\[ X_C = \frac{1}{\omega C} \Rightarrow X_C \gg R \text{ as } \omega \rightarrow 0 \]

• If you input music, voltage across resistor would be like treble and voltage across capacitor would be like bass. Build your own speaker cross-over for woofer and tweeter.
LRC filters – Analysis

**FIGURE 36.17** A series RLC circuit.

\[ \mathcal{E} = \mathcal{E}_0 \cos \omega t \]
LRC Filters – Analysis

1. Current is the same at all points in circuit at all time. Choose an arbitrary current vector (time).

2. Phase: Resistor VR in phase with I, I in capacitor leads Vc, I in inductor lags VL
   Amplitude: For a given I peak value, we know VR, Vc and VL peak

3. At any instant in time, we have (Kirchoff’s loop law):
   \[ \vec{V}_R \cdot \hat{x} + \vec{V}_L \cdot \hat{x} + \vec{V}_C \cdot \hat{x} = \vec{E} \cdot \hat{x} \]
   Point in opposite directions
   \[ \text{Assume } |V_L| > |V_C| \]
   \[ \Rightarrow \vec{V}_R + (|V_L| - |V_C|) \vec{V}_L = \vec{E} \]
LRC Filters – Analysis

Analyzing an RLC circuit

\[ \mathbf{V}_R + (|V_L| - |V_C|) \hat{\mathbf{V}}_L = \mathbf{E} \]

\[ \mathcal{E}_0^2 = V_R^2 + (V_L - V_C)^2 = [R^2 + (X_L - X_C)^2]I^2 \quad (36.23) \]

where we wrote each of the peak voltages in terms of the peak current \( I \) and a resistance or a reactance. Consequently, the peak current in the RLC circuit is

\[ I = \frac{\mathcal{E}_0}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \quad (36.24) \]

The three peak voltages, if you need them, are then found from \( V_R = IR, V_L = IX_L \), and \( V_C = IX_C \).
RC Filters – Analysis

Analyzing an RLC circuit

\[
I = \frac{\mathcal{E}_0}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{\mathcal{E}_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}
\]

Want an "Ohm's Law" form, so let:

\[
I = \frac{\mathcal{E}_0}{Z} \Rightarrow Z, \text{ impedance } = \sqrt{R^2 + (X_L - X_C)^2}
\]

Where \( X_L = \omega L \), \( X_C = \frac{1}{\omega C} \)
LRC Filters – Analysis

\[ I = \frac{\varepsilon_0}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{\varepsilon_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \]

Consider voltage across resistor:

\[ V_R = IR = (\varepsilon_0 R) \left( \frac{1}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} \right) \]

\[ V_R \text{ is max when } \omega L = \frac{1}{\omega C}, \text{ or } \omega = \sqrt{\frac{1}{LC}} = \omega_C \]

\[ \Rightarrow V_{max} = \varepsilon_0 \]

\[ V_R \text{ decreases by increasing or decreasing } \omega \text{ away from } \omega_C \]
LRC Filters – Analysis

\[ V_R = (\varepsilon_0 R) \frac{1}{\sqrt{R^2 + \left(\frac{1}{\omega L} - \frac{1}{\omega C}\right)^2}} \]

Vary R, choose L=C=1: R={0.01, 0.02, 0.04, 0.1, 1, 5}
### Basic circuit elements

<table>
<thead>
<tr>
<th>Element</th>
<th>$i$ and $v$</th>
<th>Resistance/reactance</th>
<th>$I$ and $V$</th>
<th>Power</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistor</td>
<td>In phase</td>
<td>$R$ is fixed</td>
<td>$V = IR$</td>
<td>$V_{\text{rms}}I_{\text{rms}}$</td>
</tr>
<tr>
<td>Capacitor</td>
<td>$i$ leads $v$ by $90^\circ$</td>
<td>$X_C = 1/\omega C$</td>
<td>$V = IX_C$</td>
<td>0</td>
</tr>
<tr>
<td>Inductor</td>
<td>$i$ lags $v$ by $90^\circ$</td>
<td>$X_L = \omega L$</td>
<td>$V = IX_L$</td>
<td>0</td>
</tr>
</tbody>
</table>

#### RC filter circuits

$$V_C = \frac{\epsilon_0 X_C}{\sqrt{R^2 + X_C^2}}$$

$V_C \to \epsilon_0$ as $\omega \to 0$

A **low-pass filter** transmits low frequencies and blocks high frequencies.

$$V_R = \frac{\epsilon_0 R}{\sqrt{R^2 + X_C^2}}$$

$V_R \to \epsilon_0$ as $\omega \to \infty$

A **high-pass filter** transmits high frequencies and blocks low frequencies.
Series RLC circuits

\[ I = \mathcal{E}_0/Z \text{ where } Z \text{ is the impedance} \]

\[ Z = \sqrt{R^2 + (X_L - X_C)^2} \]

\[ V_R = IR \quad V_L = IX_L \quad V_C = IX_C \]

When \( \omega = \omega_0 = 1/\sqrt{LC} \) (the resonance frequency), the current in the circuit is a maximum \( I_{\text{max}} = \mathcal{E}_0/R \).

Vary R, choose L=C=1: \( R = \{0.01, 0.02, 0.04, 0.1, 1, 5\} \)