The center of mass dropped $\frac{L}{2}$.

$\Rightarrow$ The potential energy lost is $Mg \cdot \frac{L}{2}$

The moment of inertia of the rod is $\frac{1}{3}ML^2$.

The lost potential energy will convert to the rotational kinetic energy of the rod $= \frac{1}{2} I \omega^2$.

$\Rightarrow \quad Mg \cdot \frac{L}{2} = \frac{1}{2} I \omega^2 = \frac{1}{2} \left( \frac{1}{3}ML^2 \right) \omega^2$

$\therefore \quad \omega = \sqrt{\frac{3g}{L}}$

(b) break the rod to small segments connected by massless string with negligible length.
For mass segment 1, $(T_1 - \Delta m \cdot g)$ should provide required centripetal force.

$$T_1 - \Delta m \cdot g = \Delta m \cdot l_1 \cdot \omega^2$$

$$\Rightarrow T_1 = \Delta m \cdot l_1 \cdot \omega^2 + \Delta m \cdot g$$

For mass segment 2,

$$T_2 - T_1 - \Delta m \cdot g = \Delta m \cdot l_2 \cdot \omega^2$$

$$\Rightarrow T_2 = \Delta m \cdot l_1 \cdot \omega^2 + \Delta m \cdot l_2 \cdot \omega^2 + 2 \Delta m \cdot g$$

For mass segment 3,

$$T_3 - T_2 - \Delta m \cdot g = \Delta m \cdot l_3 \cdot \omega^2$$

$$\Rightarrow T_3 = \Delta m \cdot l_1 \cdot \omega^2 + \Delta m \cdot l_2 \cdot \omega^2 + \Delta m \cdot l_3 \cdot \omega^2 + 3 \Delta m \cdot g$$

$$\vdots$$

For mass segment $n$

$$T_n = \sum_{i=1}^{n} (l_i \omega^2 + g) \Delta m$$

when $\Delta M = \frac{M}{L} \Delta L \to 0$, $\sum_{i=1}^{n} \to \int_{0}^{L}$

$$T = \int_{0}^{L} (l \omega^2 + g)(\frac{M}{L} \Delta L)$$

$$= \frac{M}{L} \left( \frac{1}{2} \omega^2 L + gL \right)$$

$$= \frac{M}{L} \left( \frac{1}{2} \frac{3g}{L} L^2 + gL \right)$$

$$= \frac{5}{2} Mg$$

Method 2: Chapter 9, page 278.
2.

(a) 

The moment of inertia of a pulley is \( I = \frac{1}{2} MR^2 \)

The rotational kinetic energy of the pulley is

\[
K_{\text{pulley}} = \frac{1}{2} I \omega^2 = \frac{1}{2} \left( \frac{1}{2} MR^2 \right) \left( \frac{2v}{R} \right)^2 = \frac{1}{4} MV^2
\]

From energy conservation

\[
(m_2 - m_1) gh = \frac{1}{2} m_1 u^2 + \frac{1}{2} m_2 v^2 + \frac{1}{2} I \omega^2 = \frac{1}{2} (m_1 + m_2) u^2 + \frac{1}{4} MV^2
\]

\[
\therefore u = \sqrt{\frac{4(m_2 - m_1) gh}{2(m_2 + m_1) + M}}
\]

when \( M \to 0 \), \( u \) reduces to \( \sqrt{\frac{2(m_2 - m_1) gh}{m_2 + m_1}} \) as expected.

(b) block \( m_2 \) drops \( h \) under acceleration \( a \) and final reaches velocity \( v \)

\[ U^2 = 2ah \quad a = \frac{v^2}{2h} = \frac{2(m_2 - m_1) g}{2(m_2 + m_1) + M} \]
\[
\begin{aligned}
\{ & \text{For } m_1 : \quad T_1 - m_1 g = m_1 a \\
& \text{For } m_2 : \quad T_2 - m_2 g = -m_2 a \\
& \text{For Pulley : } \quad c = I \omega \Rightarrow (T_2 - T_1) R = \left[ \frac{1}{2} MR^2 \right] \frac{a}{R} \\
\Rightarrow & \quad [m_2 g - m_2 a] - (m_1 g + m_1 a) \quad R = \left[ \frac{1}{2} MR^2 \right] \frac{a}{R} \\
\therefore & \quad (m_2 - m_1) g = [(m_2 + m_1) + \frac{1}{2} M] a
\end{aligned}
\]

\[
\alpha = \frac{2(m_2 - m_1) g}{2(m_2 + m_1) + M} \quad \text{consistent with (b)}
\]
3. 

(a) 

\[ I = \frac{1}{2} M_1 R_1^2 + \pi x \frac{1}{2} M_2 R_2^2 \]

\[ = \frac{1}{2} M_1 R_1^2 + M_2 R_2^2 \]

(b) The yo-yo has two types of movement: it drops by velocity \( V \) and rotates by \( \omega \)

Because these two movement is constrained by string

\[ \therefore V = \omega R_1 \]

By energy conservation, we have

\[ (M_1 + 2M_2) gh = \frac{1}{2} (M_1 + 2M_2) V^2 + \frac{1}{2} I \omega^2 \]

\[ = \frac{1}{2} (M_1 + 2M_2) V^2 + \frac{1}{2} \left( \frac{1}{2} M_1 R_1^2 + M_2 R_2^2 \right) \left( \frac{V}{R_1} \right)^2 \]

\[ \Rightarrow V = \sqrt{\frac{4 (M_1 + 2M_2) gh}{2 (M_1 + 2M_2) + \left[ M_1 + 2M_2 \left( \frac{R_2}{R_1} \right)^2 \right]}} \]
The yo-yo drops by acceleration $a$, and rotation by $\alpha$

$$\Rightarrow \quad a = \alpha R_1$$

For dropping movement:

$$T = (M_1 + 2M_2)g = -(M_1 + 2M_2)\alpha$$

For rotation, the torque is

$$\tau = I\alpha \Rightarrow T.\vec{R}_1 = \left(\frac{1}{2}M_1R_1^2 + M_2R_2^2\right)\left(\frac{\alpha}{R_1}\right)$$

$$\Rightarrow \quad (M_1 + 2M_2)g = \left\{(M_1 + 2M_2) + \left[\frac{1}{2}M_1 + M_2\left(\frac{R_2}{R_1}\right)^2\right]\right\} \alpha$$

$$\Rightarrow \quad \alpha = \frac{2(M_1 + 2M_2)g}{2(M_1 + 2M_2) + \left[M_1 + 2M_2\left(\frac{R_2}{R_1}\right)^2\right]}$$

This acceleration is achieved by dropping $h$.

$$\Rightarrow \quad \omega^2 = 2\alpha h$$

$$\omega = \sqrt{\frac{4(M_1 + 2M_2)g h}{2(M_1 + 2M_2) + \left[M_1 + 2M_2\left(\frac{R_2}{R_1}\right)^2\right]}}$$

Agree with (b)
Assume the friction force between block and cylinder is \( f \). The acceleration of cylinder relative to ground is \( a_c \). The acceleration of block relative to ground is \( a_b \).

Using Newton's law on block,
\[
F - f = ma_b
\]

Using Newton's law on cylinder,
\[
f = Ma_c
\]

From the torque of the cylinder about center of mass of cylinder we know
\[
\tau = fR = I \alpha = \left(\frac{1}{2}MR^2\right) \alpha
\]

Because of non-slipping condition, we have
\[
a_b - a_c = \alpha R \Rightarrow fR = \left(\frac{1}{2}MR^2\right) \frac{a_b - a_c}{R}
\]

\[
\Rightarrow a_b = 3a_c
\]

\[
a_b = \frac{3F}{3m+M}
\]
The normal force between the sphere and the incline is

\[ f_s = Mg \cos \theta \mu_s \]  \hspace{1cm} (1)

The downward linear movement gives

\[ Mg \sin \theta - f_s = Ma \]  \hspace{1cm} (2)

The torque of the sphere and the non-slipping condition give

\[ \tau = I \alpha \]

\[ \Rightarrow f_s R = \left( \frac{2}{5} MR^2 \right) \frac{a}{R} \]  \hspace{1cm} (3)

Combine (1), (2), (3)

\[ Mg \sin \theta - Mg \cos \theta \mu_s = Ma \]

\[ a = \frac{5}{2} g \cos \theta \mu_s \]

\[ \Rightarrow \sin \theta - \cos \theta \mu_s = \frac{5}{2} \cos \theta \mu_s \]

\[ \tan \theta = \frac{7}{2} \mu_s \]

\[ \theta = \tan^{-1} \left( \frac{7}{2} \mu_s \right) \]