SPEED OF TRANSVERSE PULSE ON A STRETCHED STRING, PERIODIC WAVE, ENERGY TRANSPORT

We now know that we can describe a transverse periodic wave of wavelength $\lambda$ and frequency $f$ by the equation $y = A \sin(kx - \omega t)$ with $k = \frac{2\pi}{\lambda}$ and $\omega = 2\pi f$ while $A \perp \hat{x}$ with $\omega = vk$ [same as $v = \lambda f$].

To generate a “pulse” we need to sum up many, many periodic waves with different wavelengths, frequencies and amplitudes. Experimentally, all we need is to take a string of length $L$ and mass $m$ tie its one end, pass the other over a pulley and hang a mass $M$.

We define linear density $\mu = \frac{m}{L}$

The string will develop tension $F = (Mg)$ everywhere. We will make the string very long, so we do not need to worry about what happens at the ends as of yet. If we “tweak” it, we can observe a pulse such as shown below traveling along it.
We will keep the amplitude small. Let us concentrate on a small piece of length $\Delta x$ and ask what happens when the pulse comes along. As is clear $\Delta x$ is lying there minding its own business when the pulse arrives and $\Delta x$ must travel on a curved path to participate in the passage of the pulse. Indeed we can imagine that $\Delta x$ is carried around a circle of radius $R$ at speed $\nu$.

Since $\Delta x$ has a mass of $\rho \Delta x$ it needs a force $F = -\frac{\Delta x v^2 R}{R}$ to go around the circle. The question is, what force is available to make this happen. Let us make $\Delta x$ big and draw forces:

Immediately, we see that the net force
Along $x$ (parallel to string) is zero. But the $y$-components due to the tension add
So available force at $P$ is $F_A = -2F_y \hat{y}$

$$= -2F \sin \theta \hat{y}$$

$$\approx -2F \theta \hat{y}$$

Since $\theta \ll 1$.

$$-2F \frac{\Delta x}{2R} \hat{y}$$

$$= -F \frac{\Delta x}{R} \hat{y}$$

While at $P$ the needed $F$ is $\frac{\mu \Delta x v^2}{R} \hat{y}$. If $F_A = F_{\alpha \Delta x}$ can happily participate in the pulse. That is, we must require $\frac{F \Delta x}{R} = \frac{\mu \Delta x v^2}{R}$. So $v = \sqrt{\frac{F}{\mu}}$ is the speed of a small amplitude pulse in a string which has a tension $F$ in it and a linear density (mass per unit length) of $\mu \ kg/m$. It seems reasonable that for a periodic wave on our string we can write

$$y = A \sin(\omega x - \omega t)$$

$$\omega = \nu k$$

$$\nu = \frac{F}{\sqrt{\mu}}$$

Provided that we keep $A \ll \lambda$ so all angles are small [we needed $\theta \ll 1$ in our proof].

\[ \textbf{Note that when } \omega \ll 1, \ \sin \theta \approx \theta \]

$$\theta = \frac{S}{R}$$

$$\omega \approx \frac{h}{a}$$

$$\omega \ll 1$$

$$\frac{0}{a} \approx \frac{0}{h} \approx \frac{S}{R}$$
ENERGY TRANSPORT BY SINE WAVE ON STRING

Every point on the string has a \( y \) coordinate which varies as \( y = A \sin \omega t \). This is like linear harmonic motion so every point has a transverse velocity

\[ v_y = A \omega \cos \omega t \]

A unit length of string will therefore have a kinetic energy \( K = \frac{1}{2} \mu A^2 \omega^2 \cos^2 \omega t \)

Whose maximum value (which is equal to total energy, KIN + POTL) will be \( K_{\max} = \frac{1}{2} \mu A^2 \omega^2 \)

wave travels by \( v \) m/s so energy transport per second \( \eta = \frac{1}{2} \mu A^2 \omega^2 v \)

Since \( F = \mu v^2 \) we can also write \( \eta = \frac{1}{2} A^2 \omega^2 \frac{F}{v} \)

\[
\begin{align*}
\text{TREAT A UNIT LENGTH OF STRING AS A "SPRING-MASS" OSCILLATOR} \\
\text{with spring constant } k_0. \\
\text{Kin. energy } K = \frac{1}{2} k_0 A^2 \omega^2 \cos^2 \omega t. \\
\text{Pot. energy } U = \frac{1}{2} k_0 A^2 \sin^2 \omega t. \\
\text{But } \omega = \sqrt{\frac{k_0}{\mu}} \\
\text{So } U = \frac{1}{2} \mu A^2 \omega^2 \sin^2 \omega t. \\
\text{Next, averaged over time } \langle \sin^2 \omega t \rangle = \langle \cos^2 \omega t \rangle = \frac{1}{2} \\
\text{So } \langle K \rangle + \langle U \rangle = \left( \frac{1}{4} + \frac{1}{4} \right) \mu A^2 \omega^2 = \frac{1}{2} \mu A^2 \omega^2 \\
\text{[of course, } \sin^2 \omega t + \cos^2 \omega t = 1 \text{ so it is not surprising } \langle K \rangle + \langle U \rangle = K + U = K_{\max} = U_{\max} \text{ since our oscillator has no friction, total energy is constant].}
\end{align*}
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