Homework #9 — Phys623 — Spring 1999
Deadline: 5 p.m., Friday, April 16, 1999.
Turn in homework in the class or put in
the box on the door of Phys 2314 by 5 p.m.

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Do not forget to write your name and the homework number!
Equation numbers with the period, like (3.25), refer to the equations of Schwabl.
Equation numbers without period, like (5), refer to the equations of this homework.

Time-Dependent Phenomena (Chapters 16.1–16.3)

1. [**5 points**] Schwabl's Problem 16.1.

   Directions: Solve this problem using the first-order perturbation theory (Ch. 16.3.2). Consider also the limit $\tau \to 0$?

2. [**5 points**] Schwabl's Problem 16.2.

   Directions: The energy eigenfunctions of an oscillator in an applied electric field are the same as the those without electric field, but displaced by a certain distance. Conversely, the original ground-state wave function $\psi_0(x-l)$ is displaced relative to the new energy basis $\psi_n(x)$. The coefficients of expansion of $\psi_0(x-l)$ in terms of $\psi_n(x)$ are discussed in Ch. 3.1.4, "Coherent States", without using Hermite polynomials.

3. [**5 points**] Schwabl's Problem 16.3

   Directions: In this problem, an atom has one electron, and the nuclear charge suddenly changes from Z to $Z+1$. Calculate the probability the electron remains in the ground state 1s and the probability of transition to the 2s state. Don't calculate the probability for 3s state.

4. A quantum particle is in an eigenstate $\psi_0(\mathbf{r})$ of Hamiltonian $\hat{H}_0$ with the energy E_0:

$$\hat{H}_0 = \frac{\hat{\mathbf{p}}^2}{2m} + V_0(\mathbf{r}), \qquad \hat{H}_0\psi_0(\mathbf{r}) = E_0\psi_0(\mathbf{r}). \tag{1}$$

   For example, you may think of an electron exposed to the electric potential of a proton in the ground state of a hydrogen atom.

   Now, let us consider a problem, where the potential V moves with a constant velocity $\mathbf{v}$:

$$V(\mathbf{r},t) = V_0(\mathbf{r}-\mathbf{v}t). \tag{2}$$

   For example, the proton of the hydrogen atom may move with the velocity $\mathbf{v}$.

   (a) [**5 points**] Using the function $\psi_0(\mathbf{r})$, construct a solution of the time-dependent Schrödinger equation **(see Hints)**:

$$i\hbar\frac{\partial\psi(\mathbf{r},t)}{\partial t} = \hat{H}(t)\psi(\mathbf{r},t), \qquad \hat{H}(t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial\mathbf{r}^2} + V_0(\mathbf{r}-\mathbf{v}t). \tag{3}$$

   (b) [**3 points**] Compare your result with the rules of the Galilean transformation in classical mechanics:

$$\mathbf{r} = \mathbf{r}' + \mathbf{v}t, \tag{4}$$
$$t = t', \tag{5}$$
$$\mathbf{p} = \mathbf{p}' + m\mathbf{v}, \tag{6}$$
$$E = E' + \mathbf{v}\mathbf{p}' + \frac{m\mathbf{v}^2}{2}, \tag{7}$$

   where the primed variables refer to the reference frame moving with the velocity $\mathbf{v}$ relative to the laboratory reference frame.

5. *Adapted from Qualifier, Fall 1994, II-2.*

Consider decay of an unstable compound nucleus $^2\mathrm{He}^5$ via the following reaction: $^2\mathrm{He}^5 \to {}^2\mathrm{He}^4 + \mathrm{n}$. (The nucleus $^2\mathrm{He}^5$, where 2 denotes the charge of the nucleus and 5 denotes the total number of protons and neutrons in the nucleus, is produced in thermonuclear fusion of deuterium $^1\mathrm{H}^2$ and tritium $^1\mathrm{H}^3$.)

Once the neutron suddenly leaves the parent nucleus $^2\mathrm{He}^5$, the daughter nucleus $^2\mathrm{He}^4$ starts to move (non-relativistically) with a velocity v due to recoil. Suppose, before the decay the nucleus $^2\mathrm{He}^5$ had one electron bound in the 1s ($|100\rangle$) state.

- **(a)** [**5 points**] Calculate the probability that the electron will remain in the 1s state of the moving nucleus $^2\mathrm{He}^4$.
- **(b)** [**3 points**] Find the formulas for the probabilities of the electron excitation to the following states of the moving nucleus $^2\mathrm{He}^4$: $|210\rangle$, $|200\rangle$, $|211\rangle$. Qualitatively sketch the probabilities as functions of v. How do they behave for small v, large v? what are the symmetry restrictions? To what power of v are the probabilities proportional at small v? Do not calculate the integrals in this part.

6. *Adapted from Qualifier, September 1992, II-2.*

At the time $t < 0$, a system is in a state $|1\rangle$ which has the same energy as a state $|2\rangle$. At the time $t = 0$, a perturbation V, which mixes the states $|1\rangle$ and $|2\rangle$, is suddenly turned on and remains constant for $t > 0$.

- **(a)** [**5 points**] Find exactly the probability $W(t)$ to find the system in the state $|2\rangle$ as a function of time t.
- **(b)** [**3 points**] Verify that in the limit of small t the result agrees with the first order of the perturbation theory.
- **(c)** [**2 points**] For which times t does the probability $W(t)$ vanish?

7. *The opposite limiting case to the sudden perturbation is a very slow, adiabatic perturbation. If the system is initially in an energy eigenstate of a discrete spectrum, the system remains in that state (does not make transitions to other states of the spectrum); however, the states itself gradually evolves into something different from what it was originally. This approximation is valid if the characteristic frequency of the perturbation, ω, is much smaller than the distance between the energy levels, ΔE: $\hbar\omega \ll \Delta E$. Adiabatic approximation is discussed in detail in Ch. 10 of the book by Griffiths.*

[**5 points**] A free two-dimensional rotator (see Problem 5 of Homework 1), which has the moment of inertia I and dipole moment d, is in its ground state at the time $t < 0$. At the time $t > 0$, a homogeneous electric field $\vec{\mathcal{E}}(t) = \mathcal{E}(t)\vec{n}$ is gradually turned on as $\mathcal{E}(t) = \mathcal{E}_0[1 - \exp(-t/\tau)]$. The following conditions are satisfied:

$$\hbar^2/I \ll d\mathcal{E}_0 \ll \tau\hbar^3/I^2, \tag{8}$$

which means that the field is strong, but is turned on slowly. Find the probability distribution of the angular momentum L perpendicular to the rotation plane at $t \to +\infty$.

8. [**5 points**] Schwabl's Problem 16.10. Assume that the Hamiltonian has the form $\hat{H} = p^2/2m + V(x)$.

Hints

4a Let us go the reference frame that moves with the velocity $\mathbf{v}$. In this reference frame, the potential V does not move, thus ψ_0 is a solution of the Schrödinger equation. Going back to the laboratory reference frame, one concludes that the probability density should move together with the potential as $|\psi_0(\mathbf{r} - \mathbf{v}t)|^2$. Thus, it is tempting to say that $\psi(\mathbf{r}, t) = \psi_0(\mathbf{r} - \mathbf{v}t)$. That is not quite right. The correct relation is:

$$\psi(\mathbf{r}, t) = e^{\varphi(\mathbf{r},t)}\, \psi_0(\mathbf{r} - \mathbf{v}t), \tag{9}$$

where $\varphi(\mathbf{r}, t)$ is a phase. To find this factor, substitute the ansatz (9) into Eq. (3) and change variables $\mathbf{r}$ and t to $\mathbf{r}' = \mathbf{r} - \mathbf{v}t$ and t.