Homework #7 — Phys623 — Spring 1999
Deadline: 5 p.m., Friday, April 2, 1999.
Turn in homework in the class or put in
the box on the door of Phys 2314 by 5 p.m.

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Do not forget to write your name and the homework number!
Equation numbers with the period, like (3.25), refer to the equations of Schwabl.
Equation numbers without period, like (5), refer to the equations of this homework.

The Zeeman Effect and the Stark Effect (Chapter 14)

§76 "An atom in an electric field" and §113 "An atom in a magnetic field" from the book by Landau and Lifshitz are recommended.

1. [**5 points**] In section 14.1.3 Schwabl computes the Zeeman effect in hydrogen for an arbitrary magnetic field using the basis $|n, j, m_j, l\rangle$ in which the spin-orbit term is diagonal but the Zeeman term is not. Redo the analysis using the basis $|n, l, m_l, m_s\rangle$ in which conversely the Zeeman term is diagonal but the spin-orbit term is not, and check that you arrive at the same result for the energy levels.

2. **(a)** [**3 points**] Consider a system whose angular momentum $\hat{\mathbf{J}} = \hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2$ consists of two parts $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$ and whose magnetic moment is

$$\hat{\mu} = \gamma_1 \hat{\mathbf{J}}_1 + \gamma_2 \hat{\mathbf{J}}_2. \tag{1}$$

In a state $|J, J_z, J_1, J_2\rangle$, show that

$$\langle \hat{\mu}_x \rangle = \langle \hat{\mu}_y \rangle = 0, \qquad \langle \hat{\mu}_z \rangle = \gamma J_z \tag{2}$$

and calculate the coefficient γ.

   (b) [**2 points**] Show that your result reproduces the equation after Eq. (14.25) for a multielectron system with $\hat{\mathbf{J}}_1 = \hat{\mathbf{L}}$ and $\hat{\mathbf{J}}_1 = \hat{\mathbf{S}}$.

   (c) [**3 points**] Using the result of Part 2a, calculate the Zeeman effect for the singlet ($F = 0$) and the triplet ($F = 1$) hyperfine-split states of the hydrogen atom ground state (see Section 12.4.2). Assume that a magnetic field is sufficiently weak, so that the Zeeman energy splitting is small compared to the hyperfine energy splitting. Sketch the resulting energy spectrum taking into account that the magnetic moment of proton is much smaller that the magnetic moment of electron.

   (d) [**3 points**] Calculate and sketch the Zeeman energy levels for the previous problem 2c in the opposite limit where the magnetic field is sufficiently strong, so that the Zeeman energy splitting is big compared to the hyperfine energy splitting.

3. *Adapted from Qualifier, Spring 1991, II-2.*

   (a) [**3 points**] Using perturbation theory in the lowest non-vanishing order in B, calculate the change of the ground state energy of the He atom in a magnetic field B. Use the variational wave function found in Section 13.2.3. Is this change of the energy positive or negative? (Chapter 7.2 may be useful.)

   From this energy shift, find magnetic susceptibility of the He atom in the ground state. Is the susceptibility paramagnetic or diamagnetic?

   (b) [**3 points**] Estimate magnetic susceptibility of the He atom in the $^3P_0$ state. Compare result with the magnetic susceptibility of the ground state found in Problem 3a.

4. **(a)** [**2 points**] From Eq. (14.29) obtain a formal expression for the polarizability α of the hydrogen atom in the ground state in terms of a sum over the states $n = 2, 3, \ldots$

(b) [**3 points**] Find a lower bound on the polarizability by keeping only the term with $n = 2$ in the sum. Express your answer as $C_l a^3 < \alpha$, where C_l is a numerical constant that you need to calculate. (Notice that polarizability, as well as magnetic susceptibility, has the dimensionality of *length*^{3} in the CGS system. Explain this!)

(c) [**5 points**] Find an upper bound on the polarizability by replacing E_n by E_2 in the denominator and taking the sum exactly. (It may be useful to treat the sum as a sum over all intermediate states except $n = 1$.) Express your answer as $\alpha < C_u a^3$, where C_u is a numerical constant that you need to calculate.

The sum (14.29) can be taken exactly using a trick invented by Dalgarno and Lewis (see the reference in Schwabl's Problem 14.1) The exact solution can be found on p. 462 of "Principles of Quantum Mechanics" by R. Shankar (second edition) and in Problem 4 of § 76 in Landau and Lifshitz.

5. [**5 points**] Calculate the energy shift of the ground state of the hydrogen atom in an electric field $\boldsymbol{\mathcal{E}}$ by variational method using the variational wave function

$$\psi_\beta = C\psi_0(r)(1 + \beta\,\mathbf{n}\cdot\mathbf{r}). \tag{3}$$

Here $\psi_0(r)$ is the ground-state wave function of the hydrogen atom, C is a normalization constant, β is a variational parameter, and $\mathbf{n}$ is a unit vector along $\boldsymbol{\mathcal{E}}$. From the energy shift, find polarizability and compare with the exact result.

6. [**3 points**] Consider the Stark effect for the $2s_{1/2}$ and $2p_{1/2}$ levels of hydrogen for an electric field $\mathcal{E}$ sufficiently weak so that the perturbation is small compared to the fine structure, but take the Lamb shift into account. What are the energy shifts? When are they approximately linear in $\mathcal{E}$, and when are they approximately quadratic?

Hints

2a See Eqs. (14.22) and (14.23).