

Homework #4 — Phys623 — Spring 1999
Deadline: 5 p.m., Wednesday, March 10, 1999.
Turn in homework in the class or put in
the box on the door of Phys 2314 by 5 p.m.

Victor Yakovenko, Assistant Professor
Office: Physics 2314
Phone: (301)–405–6151
E-mail: yakovenk@physics.umd.edu

Do not forget to write your name and the homework number!

Equation numbers with the period, like (3.25), refer to the equations of Schwabl.
Equation numbers without period, like (5), refer to the equations of this homework.

Identical Particles (Chapter 13.1)

1. Consider two particles in one dimension. One particle occupies a state described by a wave function $\psi_1(x)$; the other particle the state described by a wave function $\psi_2(x)$. The wave functions $\psi_1(x)$ and $\psi_2(x)$ are normalized, orthogonal, and have opposite parities:

$$\psi_1(-x) = \psi_1(x), \quad \psi_2(-x) = -\psi_2(x). \quad (1)$$

($\psi_1(x)$ and $\psi_2(x)$ could be the first and the second energy eigenfunctions of an oscillator.)

Answer the following questions in the cases where the particles are

- i. not identical,
- ii. identical bosons,
- iii. identical fermions.

Express your answers in terms of some integrals of $\psi_1(x)$ and $\psi_2(x)$.

- (a) [**1 points**] Write the normalized wave function of the system, $\psi(x_1, x_2)$, in terms of $\psi_1(x)$ and $\psi_2(x)$.
- (b) [**3 points**] Integrating the wave function $\psi(x_1, x_2)$ found in Problem 1a, calculate the probabilities that
- i. both particles are located in the right semi-space, $x > 0$;
 - ii. both particles are located in the left semi-space, $x < 0$;
 - iii. the particles are located in different semi-spaces.

Make sure that these three probabilities add up to unity. What is the difference in the behavior of bosons and fermions?

- (c) [**3 points**] Using the solution of Problem 1b, calculate the probability to find one particle in the
- i. right semi-space, $x > 0$,
 - ii. left semi-space, $x < 0$,

whereas we don't care where the other particle is. Do these two probabilities add up to unity?

- (d) [**3 points**] Calculate the probability density $P(l) dl$ to find the two particles at a distance l from each other.

2. [**5 points**] Two identical particles of mass m interact with each other via the three-dimensional harmonic potential $V(\mathbf{r}_1 - \mathbf{r}_2) = k(\mathbf{r}_1 - \mathbf{r}_2)^2/2$. Find the energy spectrum and the eigenfunctions of the system, when the particles are
- i. bosons,
 - ii. fermions.

3. [**5 points**] *Adapted from Physics Qualifier, Fall 1983, Problem II-1.*

Find all energy eigenvalues and eigenfunctions for a particle of mass m moving in two dimensions in a triangular box (see **Hints**):

$$U(x, y) = \begin{cases} 0, & \text{when } x > 0 \text{ and } y > 0 \text{ and } x + y < L, \\ \infty, & \text{otherwise.} \end{cases} \quad (2)$$

4. [5 points] Adapted from Qualifier, Fall 1979, II-1.

Two spin-1/2 one-dimensional fermions interact via the following potential:

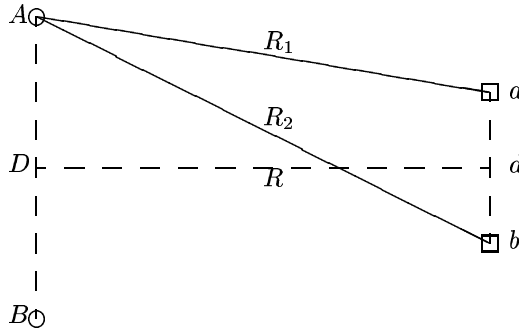
$$U(x_1 - x_2) = -\lambda\delta(x_1 - x_2)(2\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2 + \hbar^2), \quad (3)$$

where $\hat{\mathbf{S}}_1$ and $\hat{\mathbf{S}}_2$ are the spin operators of the fermions.

Can the two fermions form a bound state? Write the energy, the wave function $\psi(s_1, x_1, s_2, x_2)$ (s_1 and s_2 are the z -axis projections of the fermions' spins), and the total spin of that state. Give the answers in two cases:

- The fermions are distinguishable.
 - The fermions are indistinguishable.
5. In 1956, R. H. Brown and R. Q. Twiss suggested a new method of measuring diameters of stars. In this problem we discuss the principle of the method.

Suppose, we wish to measure the distance D between two sources of light, A and B , located at a big distance R from two detectors, a and b , located at the distance d .



The amplitude of probability that the photon emitted by the source A is registered by the detector a is $\langle a|A \rangle = ce^{ikR_1}$, where c is a constant, $k = 2\pi/\lambda$ is the wave vector of the light, and R_1 is the distance between A and a . Similarly, $\langle b|A \rangle = ce^{ikR_2}$, $\langle a|B \rangle = ce^{ikR_2}$, and $\langle b|B \rangle = ce^{ikR_1}$.

Brown and Twiss connected the detectors a and b to a coincidence counter, which registers only those events where two photon trigger both detectors simultaneously (more precisely, within a resolution time of the circuit).

- [5 points] Show that the coincidence counting rate is proportional to $2 + \cos 2k(R_2 - R_1)$ (see Hints).
 - [3 points] Explain how the result obtained in Problem 5a can be used to measure D , if R and d are known. Assume that $R \gg D, d$.
 - [1 points] Evaluate the characteristic d necessary to perform the experiment using the following values $\lambda = 5 \cdot 10^{-7}$ m, $R = 10^{17}$ m (the distance to nearest stars), and $D = 10^{11}$ m (the diameter of a star may be of the order of the distance between the Earth and the Sun).
6. [5 points] In Problem 2 of Homework 9 from Phys622, Fall 1997, we learned that the wave functions of two-dimensional electrons in the lowest Landau level in a magnetic field B can be written as

$$\psi_{0,m}(x, y) = z^m \exp\left(-\frac{eB}{4\hbar c}|z|^2\right), \quad m \geq 0, \quad (4)$$

where $z = x + iy$, x and y being the coordinates perpendicular to the magnetic field. We also learned that, if the system has a finite radius, the total number of available states (4) is limited: $0 \leq m \leq N-1$. In this problem, we do not care to normalize the wave functions.

Suppose that N electrons (fermions) occupy all available N states (4) of the lowest Landau level. (We neglect the electron spin in this problem.) Show that the wave function of the electrons has the form:

$$\psi(z_1, z_2, \dots, z_N) = \exp\left(-\frac{1}{4l^2} \sum_{i=1}^N |z_i|^2\right) \prod_{1 \leq i < j \leq N} (z_j - z_i). \quad (5)$$

To prove this statement, you need to show that

- (a) the wave function (5) contains only the wave functions of the lowest Landau level;
- (b) all available lowest Landau level states are used in the wave function (5);
- (c) the wave function (5) changes sign when two electrons are exchanged.

Comparing Eq. (5) with the Slater determinant formula (13.12) for the wave function of N fermions, we recover the expression for the so-called Vandermonde's determinant:

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_N \\ \vdots & \vdots & \dots & \vdots \\ z_1^{N-1} & z_2^{N-1} & \dots & z_N^{N-1} \end{vmatrix} = \prod_{1 \leq i < j \leq N} (z_j - z_i). \quad (6)$$

7. Adapted from *Physics Qualifier, Fall 1989, Problem II-5*.

A big number N of non-interacting, indistinguishable, spin-1/2 fermions of mass m (electrons) is constrained to a big cube of volume $V = L^3$, so that the particle density is $n = N/L^3$. As explained in Schwabl, in the ground state, all the states with $p < p_F$ and $E < E_F$ are occupied, and all the states with $p > p_F$ and $E > E_F$ are empty, where p_F is the Fermi momentum, and E_F is the Fermi energy. The electrons have the electric charge e , which is neutralized by a uniform background of positive electric charge.

- (a) [3 points] Suppose the Fermi energy slightly changes by the amount dE . That would cause the electron concentration to change by dn . Calculate the so-called energy density of states $D = dn/dE$.
- (b) [3 points] Suppose a magnetic field B is applied to the system. Find the magnetic moment μ per unit volume induced by the magnetic field and the magnetic susceptibility $\chi = \mu/B$ (called the *Pauli spin susceptibility* in this case). Take into account the Zeeman effect of the magnetic field on electrons' spins, but neglect the orbital effect of the magnetic field (the Landau levels). Express your answer in terms of the energy density of states D found in Problem 7a (see **Hints**).
- (c) [3 points] Suppose the system is placed in a weak, slowly varying in space electric potential $\phi(\mathbf{r})$. The electric potential changes the local Fermi energy by the amount $-e\phi(\mathbf{r})$. According to Problem 7a, this produces a local electric charge density $-e^2 D\phi(\mathbf{r})$. This charge feeds back into the Poisson equation governing the distribution of the electrostatic potential $\phi(\mathbf{r})$. If an external charge q is placed in the electron gas at the point $\mathbf{r} = 0$, then the Poisson equation is

$$\Delta\phi(\mathbf{r}) = 4\pi e^2 D\phi(\mathbf{r}) - 4\pi q\delta(\mathbf{r}), \quad (7)$$

where Δ is the Laplacian differential operator, the right hand side represents the electric charge concentration, and $\delta(\mathbf{r})$ is the 3D delta-function.

Solve Eq. (7) and find the characteristic length Λ over which the electron density is perturbed and beyond which the electric potential essentially vanish. This length Λ is called the *screening length*.

- (d) [1 points] In copper, $n = 8.5 \times 10^{22} \text{ cm}^{-3}$ and $m = 9.1 \times 10^{-28}$ grams. Find the numerical value of Λ for copper and compare it with the average distance between the electrons.

Hints

- 3** Map the problem onto the problem of two one-dimensional fermions moving in a box of the length L .
- 5a** Two photons may come to the detectors a and b either both from the source A , or both from the source B , or one from the source A and another from the source B . In quantum mechanics, when there are alternatives, we add the amplitudes of probabilities, when the events are indistinguishable, and the probabilities, when the events are distinguishable.
- 7b** Because of the Zeeman effect, the magnetic field increases (decreases) the Fermi energy of the electrons with the spins parallel (antiparallel) to the magnetic field. Thus, the concentrations of the spin-up and spin-down electrons become different, which produces net magnetization of the system.