Homework #4 — Phys623 — Spring 1999
Deadline: 5 p.m., Wednesday, March 3, 1999.
Turn in homework in the class or put in
the box on the door of Phys 2314 by 5 p.m.

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Do not forget to write your name and the homework number!
Equation numbers with the period, like (3.25), refer to the equations of Schwabl.
Equation numbers without period, like (5), refer to the equations of this homework.

Relativistic Corrections (Chapter 12)

1. Neutrino is a massless spin-1/2 particle. Its wave function ξ is a spinor with two components and, instead of the nonrelativistic Schrödinger equation, it satisfies the following Weyl equation:

$$i\hbar\partial_t\xi = \pm c\,(\boldsymbol{\sigma}\cdot\mathbf{p})\,\xi, \tag{1}$$

where c is the speed of light, $\mathbf{p}$ is the momentum of neutrino, and $\boldsymbol{\sigma}$ are the Pauli matrices. The two signs in Eq. (1) correspond to the two possible types of neutrino. This is a "Schrödinger equation" with Hamiltonian $\pm c\,(\boldsymbol{\sigma}\cdot\mathbf{p})$ instead of $\mathbf{p}^2/2m$. It is Lorentz covariant, because the time and space derivatives enter on equal footing at the same first order.

(a) [**3 points**] Find the energy eigenstates of (1). Is the energy spectrum bounded from below? Compare your result with the classical energy dispersion relation of a massless particle $E = c|\mathbf{p}|$. *In the quantum field theory the states with positive energy correspond to particles, whereas the states with negative energies correspond to antiparticles.*

(b) [**3 points**] Shows that the Weyl equation (1) is not parity invariant. The + (−) sign Hamiltonian describes right (left) handed neutrinos. Show that if the + sign is chosen in (1), the spin is parallel to the momentum for positive energy momentum eigenstates.

Now consider a right-handed neutrino-like particle characterized by the two-component spinor ξ and a left-handed neutrino-like particle characterized by the two-component spinor η that are coupled together by the following equations:

$$i\hbar\partial_t\xi = c(\boldsymbol{\sigma}\cdot\mathbf{p})\xi + mc^2\eta, \tag{2}$$
$$i\hbar\partial_t\eta = -c(\boldsymbol{\sigma}\cdot\mathbf{p})\eta + mc^2\xi. \tag{3}$$

(c) [**3 points**] Show that Eqs. (2) and (3) has energy eigenvalues $E = \pm\sqrt{c^2\mathbf{p}^2 + m^2c^4}$, thus m can be interpreted as the mass of the particle **(see Hints)**. Eqs. (2) and (3) are the *Dirac equation* in the *spinor* representation.

(d) [**3 points**] Show that for $\mathbf{p} = 0$, the eigenstate of Eqs. (2) and (3) with $E = mc^2$ has the property $\eta = \xi$.

(e) [**3 points**] Let us make the superpositions of ξ and η:

$$\phi = \frac{1}{\sqrt{2}}(\xi + \eta), \qquad \chi = \frac{1}{\sqrt{2}}(\xi - \eta), \tag{4}$$

and combine them into a four-component bispinor:

$$\psi = \begin{pmatrix}\phi\\ \chi\end{pmatrix}. \tag{5}$$

Show that in this, the so-called *standard* representation, the Dirac equation (2) and (3) takes the form

$$i\hbar\partial_t\psi = [c(\boldsymbol{\alpha}\cdot\mathbf{p}) + \beta mc^2]\psi, \tag{6}$$

where

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix} \tag{7}$$

are the 4×4 Dirac matrices.

In the presence of an electromagnetic field characterized by the vector potential $\mathbf{A}$ and the scalar potential Φ, Eq. (6) becomes

$$i\hbar\partial_t\psi = \left[c\boldsymbol{\alpha}\cdot\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)+\beta mc^2+e\Phi\right]\psi, \tag{8}$$

As follows from Problem 1d and Eq. (4), $\chi \ll \phi$ in the nonrelativistic limit $p \ll mc$. This allows to reduce the 4×4 Dirac equation to the 2×2 Pauli equation for the two-component spinor ϕ and find the relativistic correction as explained in the enclosed pages from *Quantum Electrodynamics* by Berestetskii, Lifshitz and Pitaevskii (vol. 4 of the *Course of Theoretical Physics* by Landau and Lifshitz).

2. [**5 points**] Schwabl's Problem 12.1.

3. [**5 points**] Schwabl's Problem 12.2.

4. [**5 points**] Using the results of Chapter 12 find the perturbed energy levels and their degeneracies due to the relativistic corrections and the hyperfine interaction for the $n = 3$ energy level of the hydrogen atom. Sketch the energy shifts in a diagram similar to Fig. 12.1 properly reflecting their signs and relative magnitudes (approximately). Make sure that the sum of the degeneracies is equal to the original degeneracy of the $n = 3$ energy level.

5. *Adapted from Qualifier, September 1988, II-1.*

   In most calculations of atomic energy levels the nucleus is taken as a positive point charge Ze. Actually, the nuclear charge is more accurately represented by a uniform charge distribution reaching to a radius $R \approx Z^{1/3}$ Fermi. (1 Fermi = 10^{-13} cm = 2×10^{-5} Bohr radius.)

   (a) [**3 points**] Find the electrostatic potential produced by a uniformly changed sphere of the charge Ze and radius R inside and outside itself.

   (b) [**3 points**] Treating the small deviation of the potential found in Problem 5a from the Coulomb potential in the first order of perturbation theory, calculate the correction to the energy of a 1s electron due to the nuclear size effect. How does this correction depend on the nuclear charge Z?

   (c) [**3 points**] How does this correction compare numerically with the fine structure, hyperfine structure, and Lamb shift corrections for the hydrogen atom?

6. [**5 points**] *Adapted from Qualifier, August 1997, II-2.*

   Suppose the Coulomb potential $-e^2/r$ in the hydrogen atom is modified to a Yukawa potential,

$$U(r) = -\frac{e^2}{r}e^{-r/b}, \tag{9}$$

   where e is the electron charge and r is the distance between the electron and the proton. Assume that the length b is much greater than the Bohr radius a.

   Determine the splitting of the degenerate levels of the hydrogen atom due to potential (9) to the lowest nontrivial order of perturbation theory in a/b. Ignore other sources of splitting discussed in Chapter 12. Restrict you consideration to the energy levels that are not very close to zero energy.

   Useful formula for the Coulomb potential: $\langle nl|r|nl\rangle = a[3n^2 - l(l+1)]/2$ (see Eq. (6.45) of Schwabl).

Hints

1c To find solutions of the eigenvalue equation

$$E\xi = c(\boldsymbol{\sigma}\cdot\mathbf{p})\xi + mc^2\eta, \tag{10}$$

$$E\eta = -c(\boldsymbol{\sigma}\cdot\mathbf{p})\eta + mc^2\xi, \tag{11}$$

find η from Eq. (10), substitute it into Eq. (11), and use Eq. (9.18b) to get rid of the Pauli matrices.