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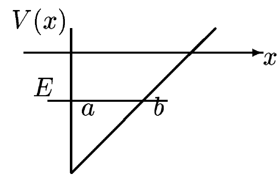
Do not forget to write your name and the homework number!
Equation numbers with the period, like (3.25), refer to the equations of Schwabl.
Equation numbers without period, like (5), refer to the equations of this homework.

The WKB Method (Chapter 11.3)

1. Study the potential of Schwabl's Problem 11.8 in the WKB approximation.



(a) [**3 points**] Show that for the potentials of this kind, which are bounded by a vertical potential wall on one side, the Bohr-Sommerfeld quantization condition takes the form

$$\int_a^b dx\, p(x) = \pi\hbar \left(n + \frac{3}{4}\right). \tag{1}$$

Directions: Take into account that the wave function must vanish at the point a and perform manipulations analogous to Eqs. (11.36) and (11.37) of Schwabl.

(b) [**2 points**] Using the quantization condition (1), find the energy levels of this potential.

2. [**5 points**] Using the Bohr-Sommerfeld quantization condition, find the energies of the bound states of the one-dimensional potential $V(x) = V_0|x/a|^\nu$, where $U_0 > 0$ and $\nu > 0$. The result may contain a numerical coefficient expressed as a definite integral, which you don't need to evaluate.

Depending on the parameter ν, how does the energy difference between the neighboring energy levels change when the quantum number n increases? Do the energy levels become denser or sparser in energy? What is the energy density of states as a function of the energy? The density of states is defined as $\Delta n/\Delta E$, where Δn is the number of states in the energy interval ΔE. Compare results with the exact solution for the harmonic oscillator when $\nu = 2$.

3. [**5 points**] Using the Bohr-Sommerfeld quantization condition, express the distance between the adjacent energy levels, $\delta E = E_{n+1} - E_n$ in terms of the frequency of classical motion in the bound state, $\omega(E)$.

4. [**3 points**] A particle stays in a bound state of the energy E. Using the Bohr-Sommerfeld quantization condition, find the change of the energy of the state δE when the potential $U(x)$ changes by a small variation $\delta U(x)$. Interpret the obtained result in terms of classical mechanics.

5. [**5 points**] Schwabl's Problem 11.6. **(see Hints)**

6. Using the quasi-classical approach, approximately calculate the total number of discrete energy levels in the following cases. All potentials V are assumed to be negative and approaching zero at infinity; that is, they have a well shape.

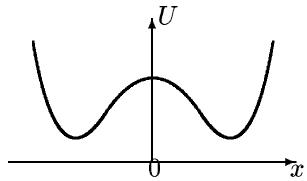
(a) [**3 points**] Generic one-dimensional potential $V(x)$.

(b) [**3 points**] Generic three-dimensional spherically-symmetric potential $V(r)$. **(see Hints)**

(c) [**5 points**] Generic three-dimensional potential $V(\mathbf{r})$ without a particular symmetry. **(see Hints)** Why the formulas are different in cases 6b and 6c?

(d) [**3 points**] Using the obtained results, prove that a three-dimensional potential $V(r)$, which decreases at infinity as $1/r^\nu$ with $0 < \nu < 2$, has an infinite number of discrete levels.

7. [**5 points**] A one-dimensional potential $U(x)$ consists of two symmetric potential wells separated by a barrier. If the barrier were impenetrable, the energy levels corresponding to the motion of the particle in one or other well would be the same for both wells. Because the passage through the barrier is possible, the energy levels split. Within the quasi-classical approximation, determine the magnitude of the splitting ΔE. **(see Hints)**



Hints

5 Classically, the particle moves on a circle of a radius R with the cyclotron frequency ω_c (7.36). Apply the Bohr-Sommerfeld quantization condition to $\int dx\, p_x$. (Distinguish canonical and kinetic momenta, see Appendix B of Schwabl).

6b Count first the number of energy levels of the radial motion with a given angular momentum l. This is, essentially, a one-dimensional problem. Then, integrate over the allowed values of l. Then, change the order of the integrals over r and l.

6c The quasi-classical quantization rule,

$$\oint p\, dx = 2\pi\hbar(n + \gamma),$$

can be interpreted in the following way. There is one quantum state ($\Delta n = 1$) per a "unit cell" of the phase space $\Delta p\, \Delta x$ of the volume $2\pi\hbar$. The number of quantum states in a given region of the phase space is proportional to the phase volume of that region (we neglect the constant γ of the order of 1, which is small compared to the big number n):

$$n = \int \frac{dp\, dx}{2\pi\hbar}\,. \tag{2}$$

Formula (2), unlike the other quasi-classical formulas, is straightforwardly generalized to a higher-dimensional case to become (in three dimensions):

$$n = \int \frac{dp_x dx}{2\pi\hbar} \frac{dp_y dy}{2\pi\hbar} \frac{dp_z dz}{2\pi\hbar}\,. \tag{3}$$

In problem 6c, you need to calculate integral (3) over the region of the phase space occupied by the bound states. For an element of the coordinate space $d^3\mathbf{r}$ at a given point $\mathbf{r}$, find the allowed range of the values of $|\mathbf{p}|$. ($|\mathbf{p}|$ cannot be arbitrarily big in a bound state.) Then, integrate over $d^3\mathbf{r}$.

7 In the WKB approximation, transmission amplitude through a one-dimensional potential barrier is given by the following expression (see Eq. (3.73)):

$$S = \exp\left(-\frac{1}{\hbar}\int_a^b |p(x)|\, dx\right), \tag{4}$$

where a and b are the turning points between which the momentum $p(x)$ is imaginary because the region is classically forbidden. The transmission rate through a barrier is equal to the amplitude S times the attempt rate $\hbar\omega/2\pi$, where ω is the frequency of classical motion in the bound state. You answer for the energy levels splitting should be expressed in terms of S and ω.