Homework #12 — Phys623 — Spring 1999
Deadline: 5 p.m., Friday, May 7, 1999.
Turn in homework in the class or put in
the box on the door of Phys 2314 by 5 p.m.

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Do not forget to write your name and the homework number!
Equation numbers with the period, like (3.25), refer to the equations of Schwabl.
Equation numbers without period, like (5), refer to the equations of this homework.

Scattering Theory (Chapter 18)

Partial Waves and Resonances

It follows from Eq. (18.27) that

$$\text{Im} f_l = k\,|f_l|^2 = k\,\sigma_l/4\pi(2l+1). \tag{1}$$

This is an analog of the optical theorem (Eq. (18.32)) for scattering in the channel with a given angular momentum l. As the optical theorem itself, Eq. (1) is a consequence of the unitarity condition for scattering in a central field (see Ch. 125 of Landau-Lifshitz). It follows from Eq. (1) that $\text{Im}(1/f_l) = -k$, thus

$$f_l = \frac{1}{g_l - ik}, \qquad \sigma_l = \frac{4\pi(2l+1)}{g_l^2 + k^2}, \qquad e^{2i\delta_l} = \frac{g_l + ik}{g_l - ik}, \tag{2}$$

where $g_l = g_l(k)$ is a real function which completely determines the scattering properties. When $g_l \to 0$, σ_l has a maximum (a resonance, Ch. 18.8); when $g_l \to \infty$, σ_l vanishes (the Ramsauer effect, Ch. 18.9).

1. [**3 points each part**] Schwabl's Problem 18.2. Don't use Schwabl's notation $g = \lambda a$ in order not to confuse with g_l of Eq. (2). In part (a), just express g_l in Eq. (2) in terms the spherical Bessel functions. In part (b), determine g_0, either from part (a), or from a fresh calculation. Don't do Schwabl's part (f); instead determine when σ_0 vanishes. Part (g): sketch $\sigma_0(k)$. You may plot and print it on computer, if you wish.

Scattering at Low Energies

At low energies, such that $ka \ll 1$, the scattering is significant only in the s-channel, with $l = 0$. Denoting $g_0(k=0) = -1/a_0$ (presuming that $g_0(k=0) \neq 0$), where a_0 is called the scattering length, we can rewrite Eq. (2) as follows:

$$f = -a_0, \qquad \sigma = 4\pi a_0^2, \qquad \delta_0 = -ka_0. \tag{3}$$

For low energies, the energy dependence of the scattering cross section (2) is

$$\sigma = \frac{2\pi\hbar^2/m}{E+\epsilon}, \qquad \text{where} \qquad \epsilon = \frac{\hbar^2}{2ma_0^2}. \tag{4}$$

When $a_0 > 0$, $-\epsilon$ is the energy of a shallow bound state, because the pole in f at $k = i/a_0$ in Eq. (2) corresponds to a bound state. When $a_0 < 0$, ϵ is called the virtual energy level and does not correspond to a bound state.

2. [**8 points**] For the potential of Problem 1, in the limit $ka \ll 1$, determine and sketch as functions of the potential strength $0 \le \lambda \le \infty$ the following quantities: a_0, f, σ, δ_0. Pay special attention to what happens to these quantities at the value of λ where a bound state appears. Check whether the Levinson theorem (18.75) is satisfied.

Born Approximation

3. [**5 points**] The criterion of validity of the Born approximation is given by the last equation of Sec. 18.5 on page 335. Let us apply this criterion to a potential of a characteristic width a and depth V_0.

Show that for slow particles, $ka \ll 1$, the criterion of validity reduces to

$$V_0 \ll \frac{\hbar^2}{ma^2}, \tag{5}$$

and for fast particles, $ka \gg 1$, to

$$V_0 \ll \frac{\hbar^2 k}{ma}. \tag{6}$$

Thus, the Born approximation is applicable either when the potential is sufficiently weak (5) (the potential well is shallow), or when the particle is sufficiently fast (6).

4. [**3 points**] According to Eq. (18.34), in the Born approximation, the amplitude of scattering, f_B, and the total cross section of scattering, σ_B, are

$$f_B(\mathbf{q}) = -\frac{m}{2\pi\hbar^2}\int V(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}}d^3r, \qquad \sigma_B = \int |f_B|^2\,d\Omega, \tag{7}$$

where $V(\mathbf{r})$ is the scattering potential, and $\mathbf{q} = \mathbf{k}' - \mathbf{k}$ is the change of particle's wave vector in the process of scattering (often called the momentum transfer).

Obviously, $\sigma_B \neq 0$. On the other hand, $f_B(\mathbf{q})$ is real, because it is a Fourier transform of a real function $U(\mathbf{r})$, thus $\mathrm{Im} f_B(0) = 0$. How would you reconcile this with the optical theorem, which says

$$\sigma = \frac{4\pi}{k}\mathrm{Im} f(0)? \tag{8}$$

5. **(a)** [**5 points**] In the Born approximation, calculate the differential and total cross sections of scattering on the potential of Problem 1. (The latter integral cannot be taken analytically.)

Formulate the conditions of applicability of the approximation.

Useful trick: In the Born approximation, the amplitude of scattering is always a function of $\mathbf{q} = \mathbf{k}' - \mathbf{k}$. Taking into account that $q^2 = 2k^2(1-\cos\theta)$, where θ is the angle of scattering, we find for the differentials: $d(\cos\theta) = -d(q^2)/2k^2$. Thus the integral over the solid angle of scattering, $d\Omega = \sin\theta\,d\theta\,d\phi = -d(\cos\theta)\,d\phi$, can be written as:

$$\sigma = \int |f(q)|^2\,d\Omega = \frac{2\pi}{2k^2}\int_0^{2k^2} d(q^2)\,|f(q)|^2. \tag{9}$$

(b) [**3 points**] Find the total scattering cross section in the limiting cases of low and high energies. Compare the result with the answer obtained in Problem 2 in the common domain of applicability of the low-energy and Born approximations.

6. **(a)** [**3 points**] A flux of j particles per unit area per unit time scatters on a potential. The particles have the initial momentum p and the mass m.

Express the force that the scattering particles exert on the potential in terms of the so-called *transport cross section*:

$$\sigma_{\mathrm{tr}} = \int (1-\cos\theta)\,d\sigma, \tag{10}$$

where $d\sigma$ is the differential cross section of scattering. (Force is the change of momentum per unit time.) *The transport cross section determines, for example, electrical resistance of a metal due to electron scattering on impurities.*

(b) [**3 points**] Calculate the transport cross section for the potential of Problem 5a.