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Homework #11 — Phys623 — Spring 1999
Deadline: 5 p.m., Friday, April 30, 1999.
Turn in homework in the class or put in the box on the door of Phys 2314 by 5 p.m.

Do not forget to write your name and the homework number!
Equation numbers with the period, like (3.25), refer to the equations of Schwabl.
Equation numbers without period, like (5), refer to the equations of this homework.

Interaction with the Electromagnetic Field (Chapter 16.4)

1. *Adapted from Qualifier, January 1998, II-2.*

An electron of mass m is subject to a spherically symmetric three-dimensional harmonic potential, so that the electron Hamiltonian is

$$\hat{H}_{\rm el} = \frac{\mathbf{p}^2}{2m} + \frac{m\Omega^2\mathbf{r}^2}{2}, \tag{1}$$

where $\mathbf{p}$ and $\mathbf{r}$ are the electron momentum and coordinate, and Ω is the frequency of the oscillator. The Hamiltonian (1) is a sum of three independent oscillator Hamiltonians in the x, y, and z directions, so the wave function of the electron can be written as a product of the corresponding wave functions.

Suppose the initial wave function of the electron is

$$|\text{initial}\rangle_{\rm el} = |0,x\rangle_{\rm el}|0,y\rangle_{\rm el}|1,z\rangle_{\rm el}, \tag{2}$$

where $|0,x\rangle$ and $|0,y\rangle$ are the ground-state wave functions of the motions along the x and y axes, and $|1,z\rangle$ is the first excited state of the motion along the z axis. By emitting a photon, the electron can make a transition to the ground state

$$|\text{final}\rangle_{\rm el} = |0,x\rangle_{\rm el}|0,y\rangle_{\rm el}|0,z\rangle_{\rm el}. \tag{3}$$

In this problem, the lifetime of the excited state (2) is studied.

(a) [**5 points**] Using the Fermi golden rule for the first-order perturbation theory, calculate the rate of spontaneous emission of photons with the wave vector $\mathbf{k}$. The transitions take place from the state $|\text{initial}\rangle_{\rm el}|0\rangle_{\rm ph}$ to the state $|\text{final}\rangle_{\rm el}|1,\mathbf{k},\lambda\rangle_{\rm ph}$. Assume that the wavelength of the emitted photon is much greater than the size of the oscillator wavefunction, so the dipole approximation may be used.

<u>Hint:</u> For calculating a matrix element, it is convenient to represent the electron momentum along the z-axis, $\hat{p}_z$, in the form

$$\hat{p}_z = -i\sqrt{\frac{\hbar\Omega m}{2}}(\hat{b} - \hat{b}^+), \tag{4}$$

where $\hat{b}^+$ and $\hat{b}$ are the raising and lowering operators for an oscillator:

$$\hat{b}^+|0,z\rangle_{\rm el} = |1,z\rangle_{\rm el}, \quad \hat{b}|1,z\rangle_{\rm el} = |0,z\rangle_{\rm el}. \tag{5}$$

(b) [**5 points**] Using the result of part 1a, or qualitative arguments, answer the following questions:

i. What is the frequency of the emitted photon?

ii. What is the polarization of a photon emitted with the wave vector $\mathbf{k}$?

iii. How does the emission rate of photons with the wave vector $\mathbf{k}$ that forms an angle θ with the z axis depend on θ? Can the system emit a photon with $\mathbf{k}$ parallel to z?

(c) [**5 points**] By integrating the result of part 1a over all possible wave vectors of the emitted photons calculate the total transition rate w and the lifetime $\tau = 1/w$ of the excited state (2). Check dimensionality and demonstrate that your result for τ has the dimensionality of time.

2. *Adapted from Qualifier, September 1993, II-2.*

Consider an empty metallic cube (the cavity) with perfectly conducting walls of length L on each side: $0 \le \{x, y, z\} \le L$. The transverse electric (TE) modes of the cavity with, for example, polarization in the z-direction, are described by a (Schrödinger picture) electric field operator of the form:

$$\vec{E} = \hat{z} E_0 \sin(N_x \pi x/L) \sin(N_y \pi y/L)(a + a^\dagger), \tag{6}$$

where N_x and N_y are integers and a and $a^\dagger$ are annihilation and creation operators. The frequency of this mode is $\omega = c\pi\sqrt{N_x^2 + N_y^2}/L$

(a) [**3 points**] Compute the expectation value of the field energy in the ground state of this mode. Set this equal to $\hbar\omega/2$ to show that in Eq. (6) E_0 is given by $E_0^2 = 8\pi\hbar\omega/L^3$. (This is a shortcut. You could instead impose the canonical commutation relations for the field operators.)

Suppose that initially there are no photons in the cavity, and there is a hydrogen atom located at a point (x_0, y_0, z_0) in a highly excited state $|nlm\rangle$ with a very large n, l equal to its maximal value $l = n - 1$, and $m = 0$. (Neglect the relativistic corrections.) An additional static magnetic field B applied along the z axis removes the degeneracy among magnetic substates. The size of the box L is selected in such a way that the frequency of the TE mode (6) is equal to the frequency of the atomic transition $n \to n' = n - 1$ with $m' = m$.

(b) [**3 points**] Show that the system (atom + radiation field) in the initial state described above and with L tuned as above can evolve via the electric dipole interaction into only one other state, and describe that state. The system is thus effectively a two-state system.

(c) [**5 points**] Using the solution of Problem 6 from Homework 9, show that the time evolution of the state of this system is oscillatory and find an expression for the oscillation frequency ν in terms of the electric dipole matrix element d and other parameters of the problem.

(d) [**5 points**] Calculate d taking into account that the radial wave function is:

$$R_{n,l=n-1}(r) \propto r^{n-1} e^{-r/na_0}. \tag{7}$$

(e) [**3 points**] Compute the frequency ν and box size L for $n = 10$, assuming that L is tuned to the lowest resonant mode $N_x = N_y = 1$ and the atom is located in the center of the cavity.

Useful information:

$$E_n = -\frac{e}{2a_0 n^2} = -\frac{13.6\,\mathrm{eV}}{n^2}$$

$$\int r^N e^{-br}\,dr = \frac{N!}{b^{N+1}}$$

$$\int d\Omega\,\cos\theta\, Y^*_{l'm'}\, Y_{lm} = \left(\frac{l^2 - m^2}{(2l+1)(2l-1)}\right)^{1/2} \delta_{m,m'}\delta_{l-1,l'}$$

3. *Adapted from Qualifier, Fall 1994, January 1998, II-5*

In this problem we explore one of the principles of cooling atoms by laser radiation (1997 Nobel Prize), the so-called Doppler cooling (there is also the so-called Sisyphus cooling).

(a) [**3 points**] Consider an atom at rest, which is illuminated by a laser plane wave of the frequency ω tuned to the transition frequency ω_0 between two energy levels of the atom: $\omega = \omega_0$.

When the atom absorbs a photon of the laser plane wave, what momentum does the photon transfer to the atom? When the atom emits a photon (as a spherical wave), does the atom acquire any momentum? Using the answers to these questions, calculate the force acting on the atom due to absorption of the photons in terms of the absorption rate,

$$R(\omega) = B\, I(\omega), \tag{8}$$

B is the Einstein coefficient, and $I(\omega)$ is the spectral density of the energy flux of the laser (see Problem 5 of Homework 10).

(b) [**5 points**] Now suppose that the atom moves with a velocity $\mathbf{v}$ and is illuminated by two laser beams shining parallel and antiparallel to $\mathbf{v}$. Using the conservation laws of energy and momentum or the Doppler shift of frequencies, calculate the frequency of a photon that the atom would need to absorb from the parallel, or alternatively from the antiparallel, beam in order to make a transition to the excited state.

Assuming that the momentum transfered by a photon is much smaller than atom's momentum mv, calculate the difference of the forces exerted on the atom by the two beams. Express you answer in terms of derivative the $dI(\omega)/d\omega$ assuming that the laser line is narrow: $\Delta\omega \ll \omega_0$, where $\Delta\omega$ the width is the laser line. Show that if the laser is tuned below the transition frequency ω_0, so that $dI(\omega)/d\omega < 0$, the net force exerted by the pair of beams is dissipative, such that

$$\mathbf{F} = -\alpha\mathbf{v}. \tag{9}$$

Find an expression for the friction coefficient α.

(c) [**3 points**] Show that illumination by three pairs of orthogonal beams decreases temperature T of the atoms and calculate the cooling rate $(dT/dt)/T$ assuming that the velocity distribution remains approximately Maxwellian.

(d) [**3 points**] Estimate the lowest temperature that can be achieved by this method. (When does the method stop to be effective?)

The Central Potential II (Chapter 17)

4. [**5 points**] Schwabl's Problem 17.2.

Don't do part (a), because we have done it already last semester; do only parts (b) and (c).

In part (b), obtain an equation for the bound state energy in terms of the spherical Bessel functions.

In part (c), take into account that a marginally bound state has the energy $E = 0$. Compare the behavior of the spherical Bessel functions in this case with the wave function given by the equation that precedes Eq. (6.13a).