Probability distribution of returns in a model with stochastic volatility

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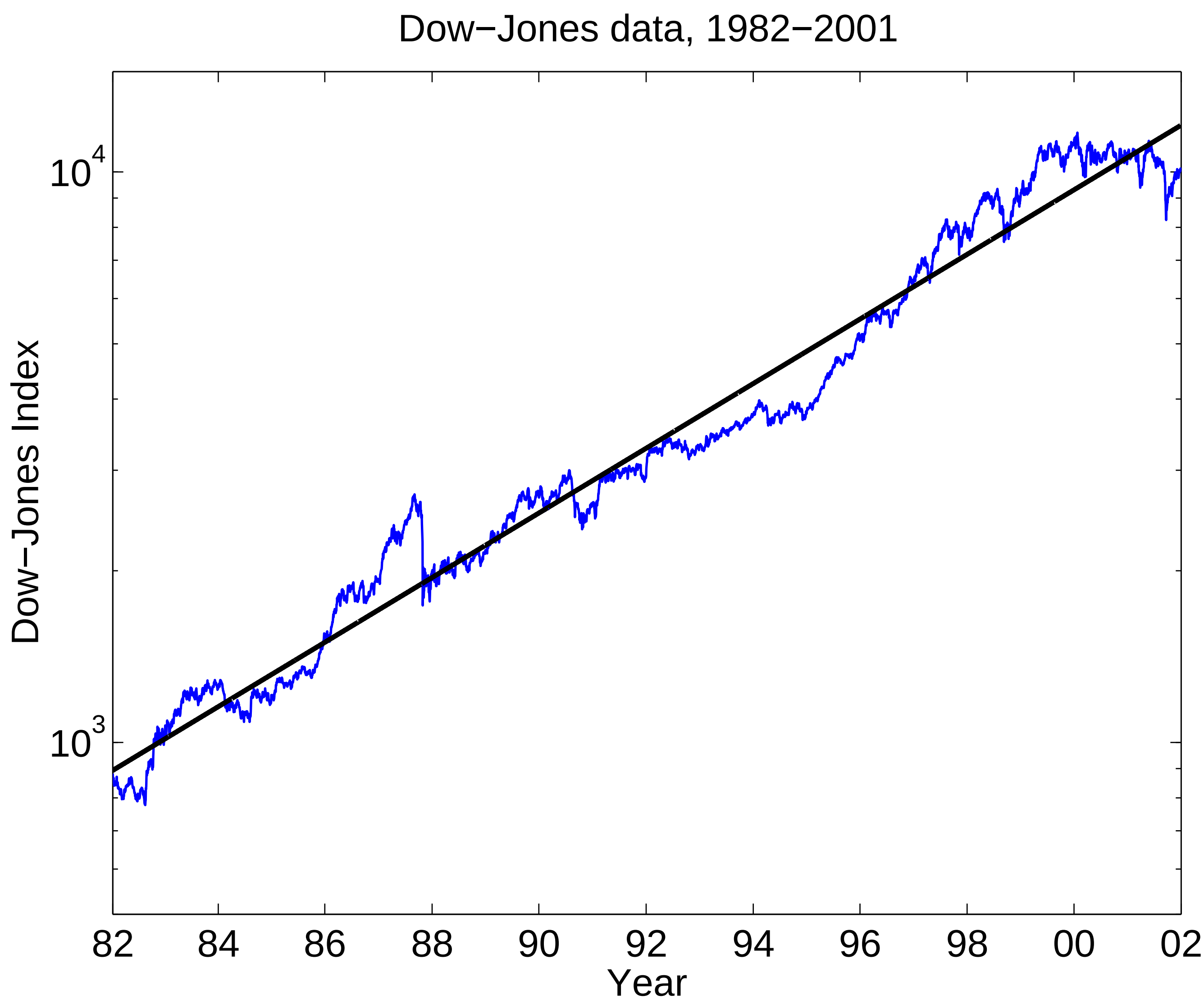
Outline

- Formulation of the problem
- Description of the model: multiplicative Brownian motion with stochastic volatility
- Exact solution of the model
- Analytical analysis in several asymptotic limits
- Comparison with the Dow-Jones data
- Conclusions

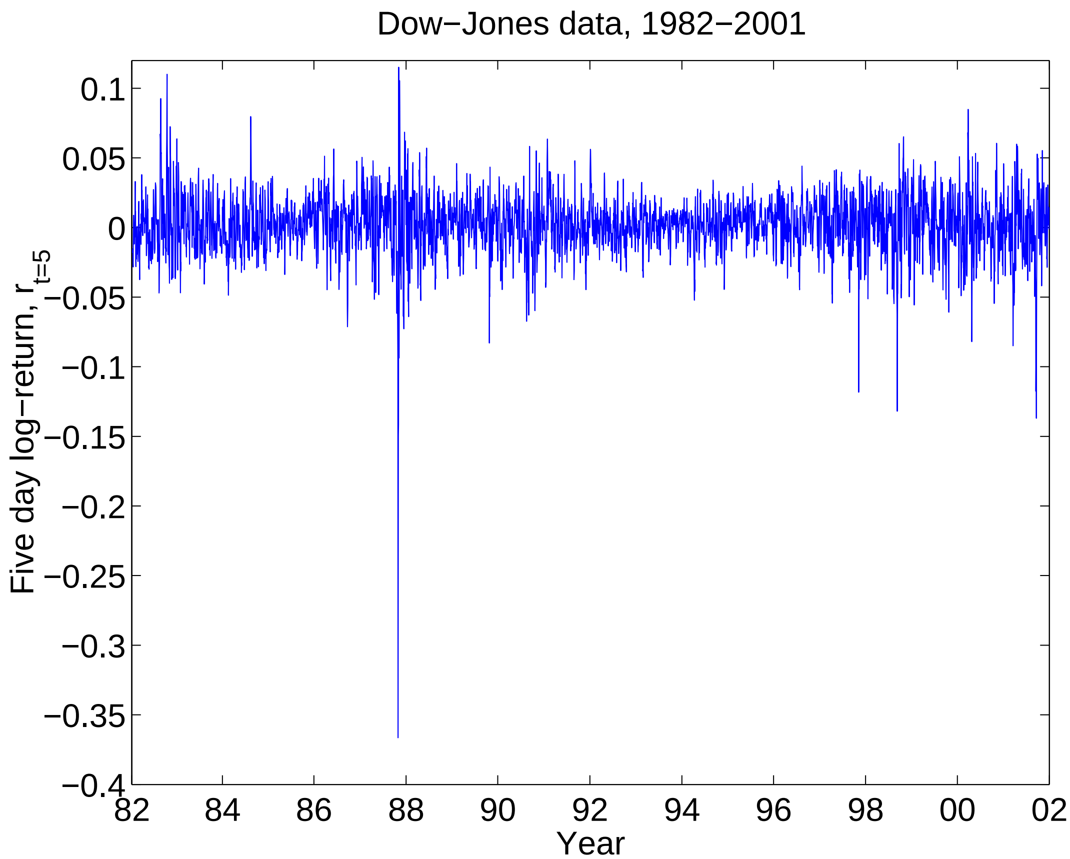


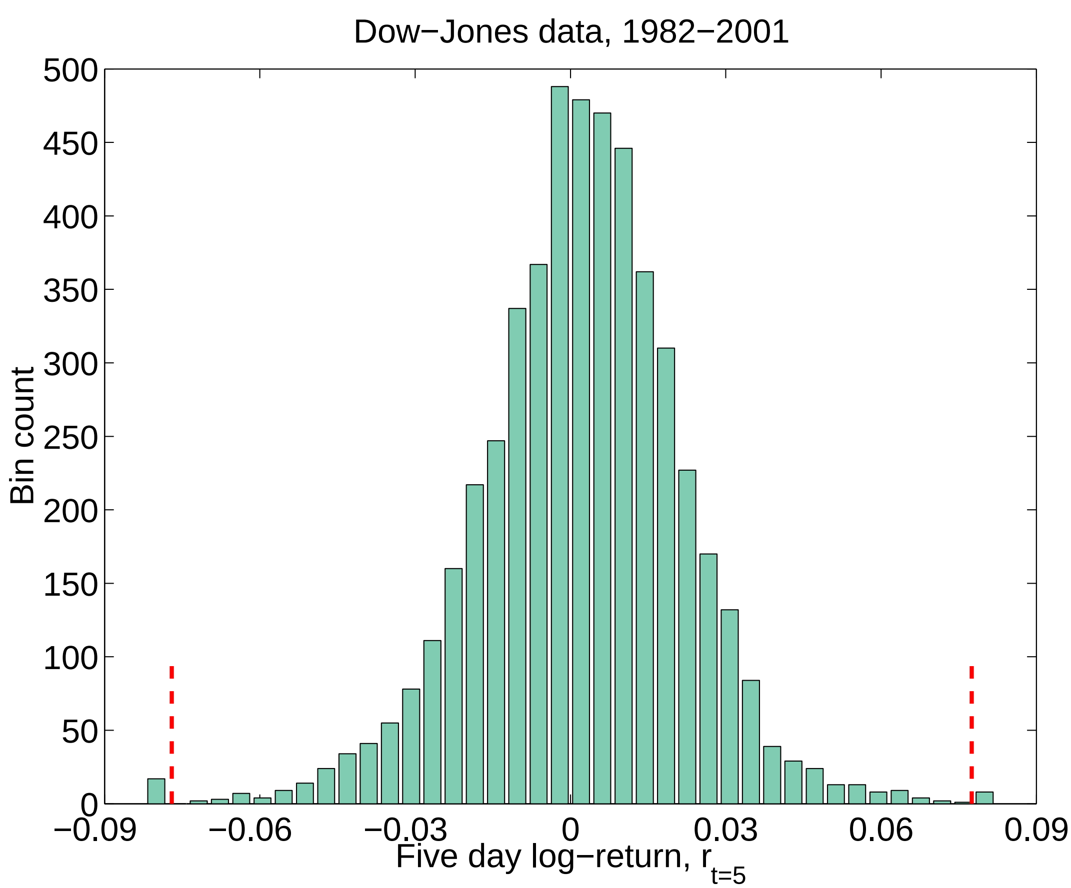
Define the log-return $r_{t_2-t_1} = \ln \frac{S_{t_2}}{S_{t_1}}$.

Average growth $\mu = \langle \frac{dr}{dt} \rangle = 13\%$ per year.

Subtracting the average growth, define the log-return fluctuations: $x_t = r_t - \mu t$.

What is $P_t(x)$, the probability to have log-return x after time lag t? $P_t(x) \to \delta(x)$ when $t \to 0$.


Dow−Jones data, 1982−2001
Five day log−return, $r_{t=5}$
0.1
0.05
0
−0.05
−0.1
−0.15
−0.2
−0.25
−0.3
−0.35
−0.4
82
84
86
88
90
92
94
96
98
00
02
Year


Dow−Jones data, 1982−2001
Bin count
500
450
400
350
300
250
200
150
100
50
0
−0.09
−0.06
−0.03
0
0.03
0.06
0.09
Five day log−return, $r_{t=5}$

Stochastic differential equation for variance: The Cox-Ingersoll-Ross or Feller process

The variance, $v_t = \sigma_t^2$, satisfies

$$dv_t = -\gamma(v_t - \theta)dt + \kappa\sqrt{v_t}\,dW_t^{(2)},$$

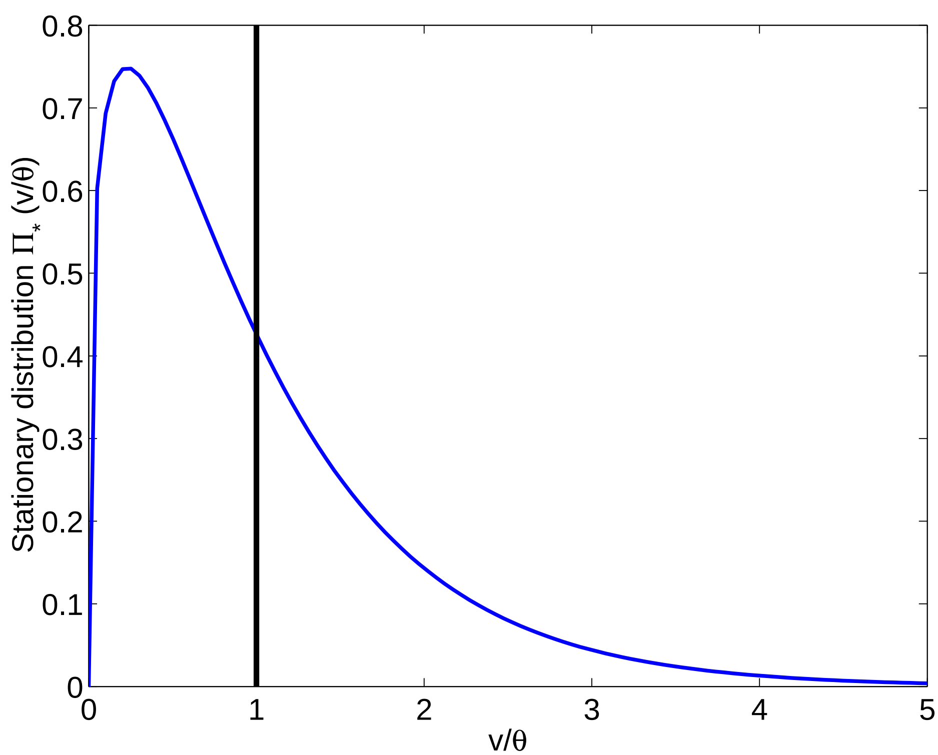
γ=relaxation rate, θ=average variance, κ=noise.

The Fokker-Planck equation for the probability distribution $\Pi_t(v)$:

$$\frac{\partial \Pi_t(v)}{\partial t} = \frac{\partial}{\partial v}\left[-\gamma(v-\theta)\Pi_t(v)\right] + \frac{\kappa^2}{2}\frac{\partial^2}{\partial^2 v}\left[v\Pi_t(v)\right].$$

The stationary distribution satisfies $\frac{\partial}{\partial t}\Pi_t(v) = 0$:

$$\Pi_*(v) \propto v^\beta \exp(-\alpha v), \quad \alpha = 2\gamma/\kappa^2, \quad \beta = \alpha\theta - 1.$$

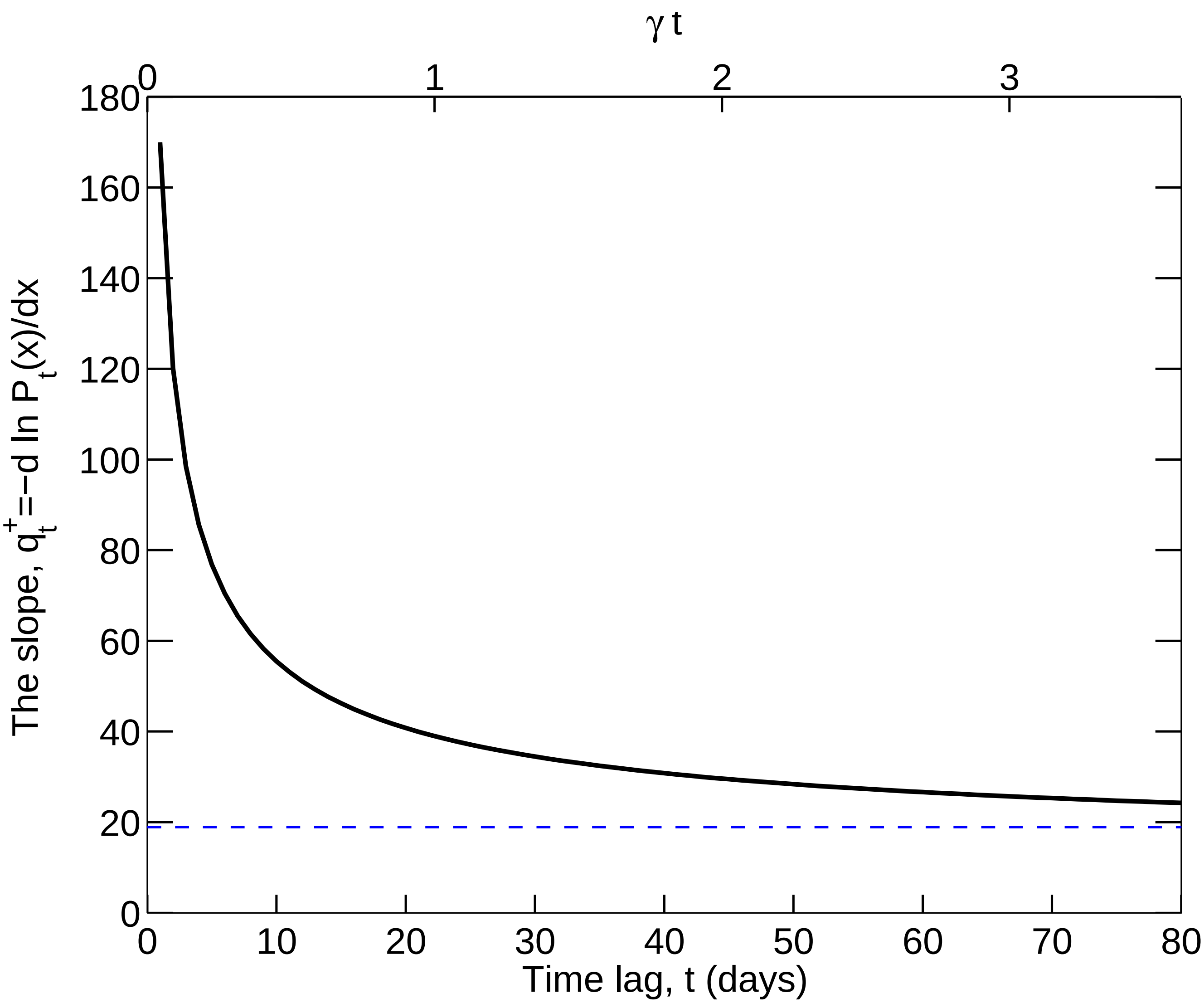


Asymptotic behavior for large log-return x

The tails of the probability distribution $P_t(x)$ are exponential in x:

$$P_t(x) \sim \begin{cases} e^{-xq_t^+}, & x > 0, \\ e^{xq_t^-}, & x < 0, \end{cases}$$

where $\pm iq_t^\pm$ are the singularities of $F_t(p_x)$ in the complex plane of p_x closest to the real axis.



Conclusions

- For the model of multiplicative Brownian motion with stochastic volatility (CIR), we calculated the probability distribution $P_t(x)$ of returns x after time lag t exactly.

- In the long-time limit $\gamma t \gg 2$, the probability distribution $P_t(x)$ has the scaling form

$$P_t(x) = P_*(z) \propto \frac{K_1(z)}{z} \quad \text{where} \quad z = a\sqrt{x^2 + (ct)^2}.$$

The relaxation time is $1/\gamma = 22$ days ≈ 1 month.

- For large log-returns x, the probability distribution $P_t(x)$ has exponential tails in x. The slopes of the tails decrease with time and then saturate when $\gamma t \gg 1$.

- We found an excellent agreement with the Dow-Jones data for 1982–2001 from $t = 1$ day to $t = 250$ days ($= 1$ year). The scaling holds for seven orders of magnitude.