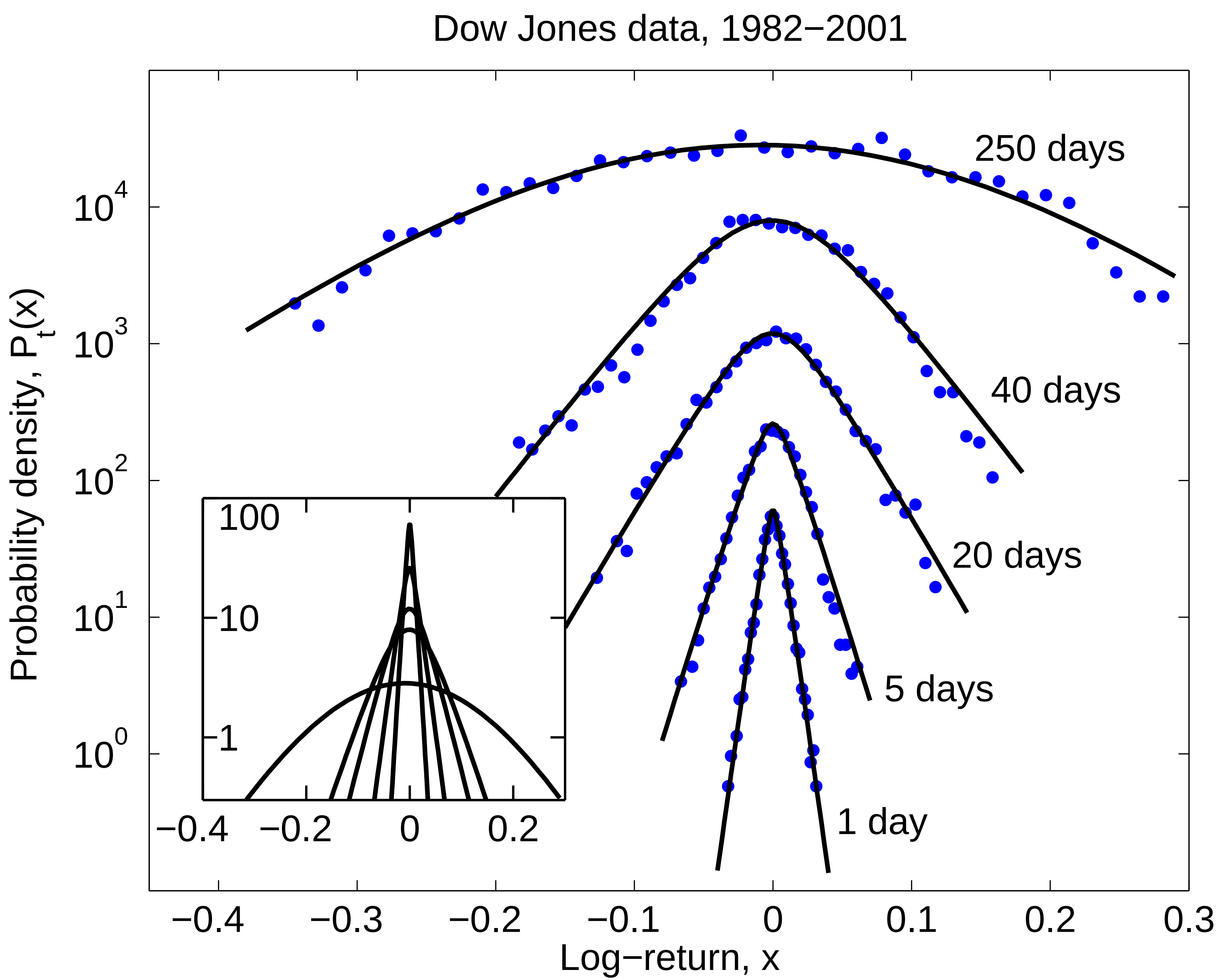
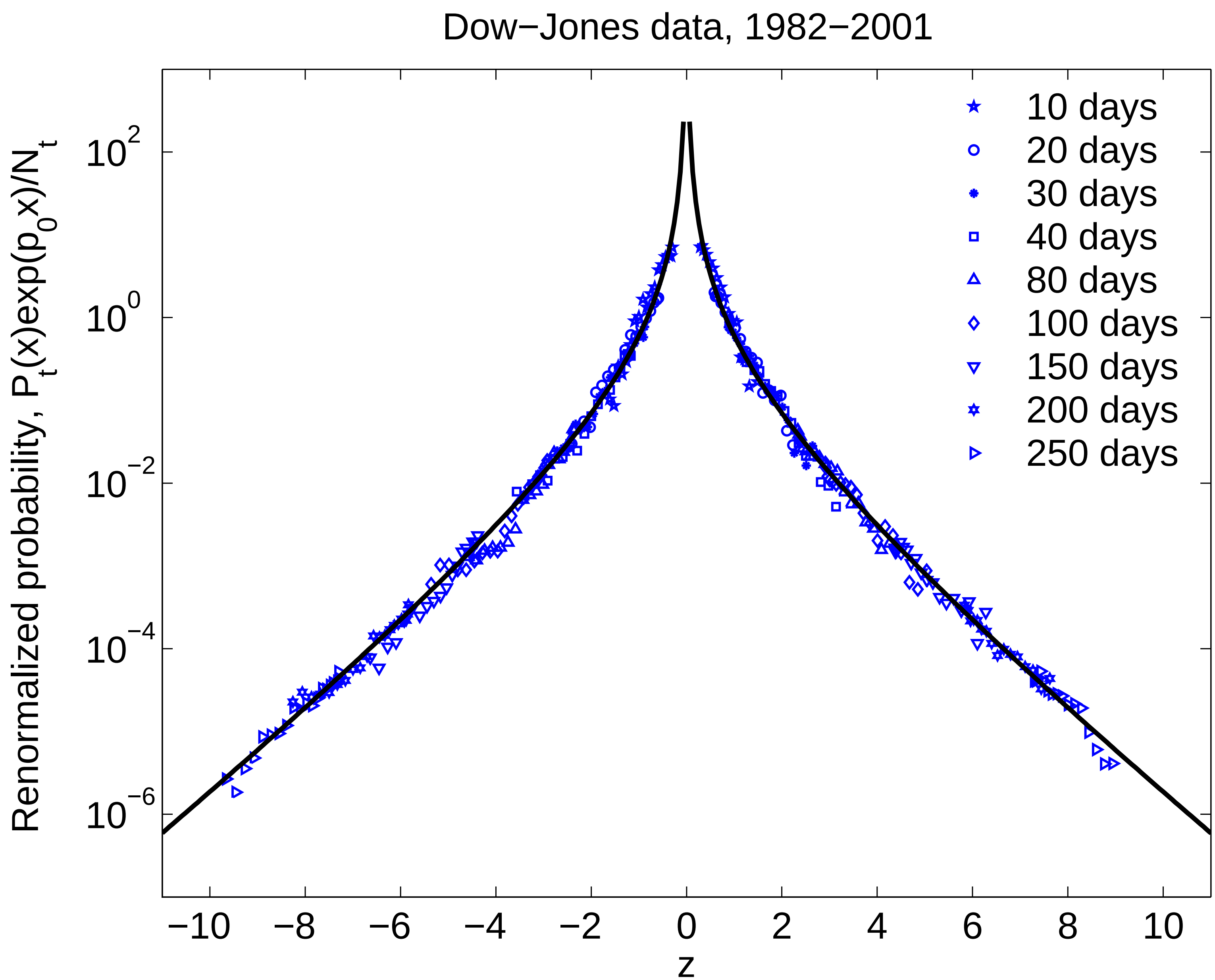


Dow Jones data, 1982−2001
250 days
40 days
20 days
5 days
1 day
10^4
10^3
10^2
10^1
10^0
Probability density, $P_t(x)$
−0.4
−0.3
−0.2
−0.1
0
0.1
0.2
0.3
Log−return, x
100
10
1
−0.4
−0.2
0
0.2

Scaling in the long-time limit, $\gamma t \gg 1$



$P_t(x) \to P_*(z) = \frac{K_1(z)}{z}$, where $z = a\sqrt{x^2 + (ct)^2}$

The simplest model: stock price S_t obeys the stochastic differential equation of a multiplicative Brownian motion

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t^{(1)},$$

where W_t is a Wiener process, σ is called volatility.

The distribution of log-returns $r_t = \ln(S_t/S_0)$ is Gaussian

$$P_t(r) = \frac{\exp\{-[r-(\mu+\sigma^2/2)t]^2\}}{\sqrt{2\pi\sigma^2 t}},$$

and the distribution of prices $P_t(S)$ is log-normal.

Shortcomings of multiplicative Brownian model:

- The tails of real data are heavier than Gaussian.
- Volatility σ is not constant, but stochastic.
- Limited success of the Black-Scholes model for option pricing.

Multiplicative Brownian motion + stochastic variance

$$\begin{cases} dx_t = -\frac{v_t}{2}\,dt + \sqrt{v_t}\,dW_t^{(1)} \\ dv_t = -\gamma(v_t - \theta)\,dt + \kappa\sqrt{v_t}\,dW_t^{(2)} \end{cases} \qquad v = \sigma^2$$

The model is truly 2D $\neq$ 1D + 1D.

The transition probability $P_t(x, v \,|\, v_i)$ satisfies the Fokker-Planck equation

$$\begin{aligned} \frac{\partial}{\partial t}P &= \frac{1}{2}\frac{\partial}{\partial x}(vP) + \gamma\frac{\partial}{\partial v}\left[(v-\theta)P\right] \\ &\quad + \frac{1}{2}\frac{\partial^2}{\partial^2 x}(vP) + \frac{\kappa^2}{2}\frac{\partial^2}{\partial^2 v}(vP) \end{aligned}$$

with the initial condition $P_{t=0}(x, v \,|\, v_i) = \delta(x)\delta(v - v_i)$.

Take two Fourier transforms

$$P_t(x, v \,|\, v_i) \to \overline{P}_{t,p_x}(v \,|\, v_i) \to \widetilde{P}_{t,p_x}(p_v \,|\, v_i).$$

The transformed PDE is of the first order

$$\left[\frac{\partial}{\partial t} + \left(\gamma p_v + \frac{i\kappa^2}{2} p_v^2 + \frac{i p_x^2 + p_x}{2}\right) \frac{\partial}{\partial p_v}\right] \widetilde{P} = -i\gamma\theta p_v \widetilde{P},$$

with the initial condition $\widetilde{P}_{t=0,p_x}(p_v \,|\, v_i) = \exp(-i p_v v_i)$.

The solution is obtained using the method of characteristics or path integrals

$$\widetilde{P}_{t,p_x}(p_v \,|\, v_i) = \exp\left(-i\tilde{p}_v(0) v_i - i\gamma\theta \int_0^t d\tau\, \tilde{p}_v(\tau)\right),$$

where the function $\tilde{p}_v(\tau)$ is the solution of the characteristic equation

$$\frac{d\tilde{p}_v(\tau)}{d\tau} = \gamma \tilde{p}_v(\tau) + \frac{i\kappa^2}{2} \tilde{p}_v^2(\tau) + \frac{i p_x^2 + p_x}{2} \quad \text{with} \quad \tilde{p}_v(t) = p_v$$

Averaging over variance

We integrate $P_t(x, v \,|\, v_i)$ over the final variance v

$$P_t(x \,|\, v_i) = \int_{-\infty}^{+\infty} dv \, P_t(x, v \,|\, v_i).$$

Assuming that v_i to has the stationary distribution $\Pi_*(v_i)$, we average over the initial variance

$$P_t(x) = \int_0^{\infty} dv_i \, \Pi_*(v_i) \, P_t(x \,|\, v_i).$$

The final result

The probability distribution of returns is given by the Fourier integral

$$P_t(x) = \int_{-\infty}^{+\infty} \frac{dp_x}{2\pi}\, e^{ip_x x + F_t(p_x)}$$

where

$$F_t(p_x) = \frac{\gamma^2\theta t}{\kappa^2} - \frac{2\gamma\theta}{\kappa^2}\ln\left[\cosh\frac{\Omega t}{2} + \frac{\Omega^2+\gamma^2}{2\gamma\Omega}\sinh\frac{\Omega t}{2}\right]$$

and the frequency $\Omega = \sqrt{\gamma^2 + \kappa^2(p_x^2 - ip_x)}$.

Asymptotic behavior for long time t

In the limit $\gamma t \gg 2$, $\quad F_t(p_x) \approx \frac{\gamma\theta t}{\kappa^2}(\gamma - \Omega).$

The Fourier integral can be taken analytically, and the probability distribution has the scaling form

$$P_t(x) = N_t\, e^{-x/2} P_*(z), \qquad P_*(z) = \frac{K_1(z)}{z},$$

where $K_1(z)$ is the first-order modified Bessel function, and

$$z = \frac{\omega_0}{\kappa}\sqrt{x^2 + \left(\frac{\gamma\theta}{\kappa}\right)^2 t^2}, \qquad \omega_0 = \sqrt{\gamma^2 + \kappa^2/4}$$

For $z \gg 1$, the Bessel function is $K_1(z) \approx e^{-z}\sqrt{\pi/2z}$,

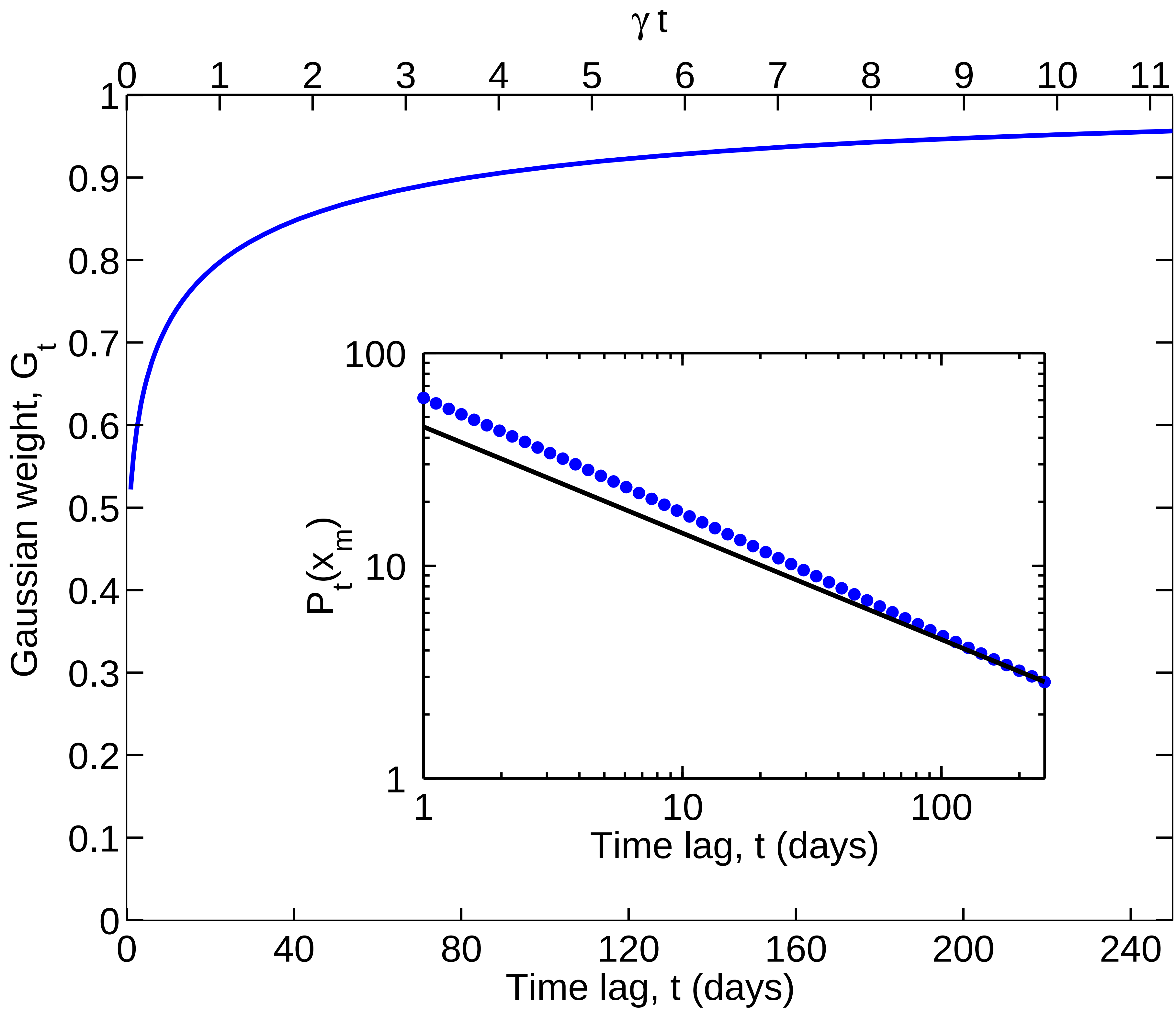
$$\ln P_t(x) \approx -\frac{x}{2} - z = -\frac{x}{2} - \frac{\omega_0}{\kappa}\sqrt{x^2 + \left(\frac{\gamma\theta}{\kappa}\right)^2 t^2}.$$

- When $|x| \gg \gamma\theta t/\kappa$, $\quad \ln P_t(x) \approx -\frac{x}{2} - \frac{\omega_0}{\kappa}|x|$.

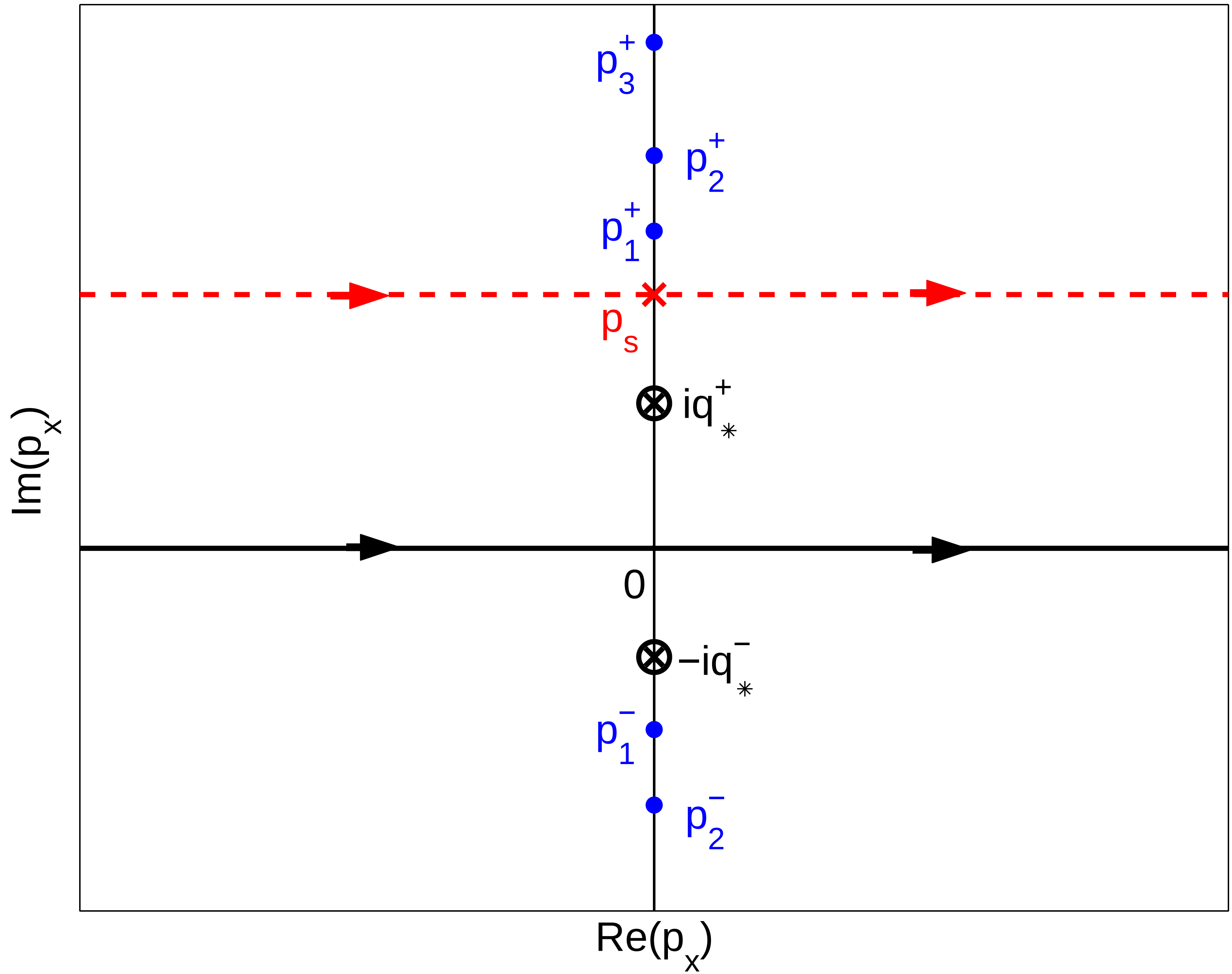
The probability distribution has exponential tails in x with time-independent slope.

- When $|x| \ll \gamma\theta t/\kappa$, $\quad \ln P_t(x) \approx -\frac{x}{2} - \frac{\omega_0}{2\gamma\theta t}x^2$,

The probability distribution for small x is Gaussian. For long time t, the probability weight in the Gaussian part tends to one.


γ t
0
1
2
3
4
5
6
7
8
9
10
11
Gaussian weight, G_t
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
40
80
120
160
200
240
Time lag, t (days)
$P_t(x_m)$
1
10
100
1
10
100
Time lag, t (days)

Singularities of $F_t(p_x)$ in the complex plane of p_x



Quantum-mechanical analogy

Define the momentum operator $\widehat{p}_v = -i\frac{d}{dv}$ conjugate to v.

The Fokker-Planck PDE is a Schrödinger equation in imaginary time

$$t\frac{\partial}{\partial t}\overline{P} = -\widehat{H}\overline{P}$$

with "Hamiltonian"

$$\widehat{H} = \frac{\kappa^2}{2}\widehat{p}_v^2\widehat{v} + i\gamma\widehat{p}_v(\widehat{v} - \theta) + \frac{p_x^2 - ip_x}{2}\widehat{v}.$$

Notice that the "Hamiltonian" $\widehat{H}$ is linear in v.

Path Integral Solution: Integrate over all trajectories

$$\overline{P}_{t,p_x}(v_f \,|\, v_i) = \int \mathcal{D}v(\tau) \int \mathcal{D}p_v(\tau) e^{S[p_v(\tau),v(\tau)]},$$

where the action $S[p_v(\tau), v(\tau)]$ for a given path is

$$S = \int_0^t d\tau \left\{ip_v(\tau)\dot{v}(\tau) - H[p_v(\tau), v(\tau)]\right\}.$$

The action is linear in $v(\tau)$. First take the integral over $\mathcal{D}v(\tau)$. The result is a delta-functional

$$\delta \left[\dot{p}_v(\tau) - \gamma p_v(\tau) - \frac{\kappa^2}{2} p_v^2(\tau) - \frac{p_x^2 - ip_x}{2}\right].$$

Taking the integral over $\mathcal{D}p_v(\tau)$ replaces $p_v(\tau)$ by $\tilde{p}_v(\tau)$, which is the solution of characteristic ordinary differential equation.