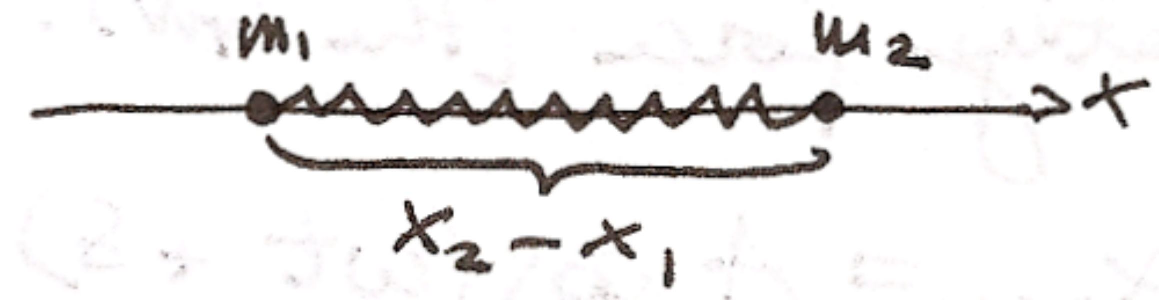
(1) Let's work in one dimension, along x. Let x_1 and x_2 be the coordinates of m_1 and m_2. Let $x_2 > x_1$.

Let's assume there's <u>no GW</u> at first. The Newtonian eqs. of motions are

$$\begin{cases} m_1 \ddot{x}_1 = +k(x_2 - x_1 - L) - b\dot{x}_1 \\ m_2 \ddot{x}_2 = -k(x_2 - x_1 - L) - b\dot{x}_2 \end{cases}$$



Sum and difference of these eqs.

$$\begin{cases} m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = -b(\dot{x}_1 + \dot{x}_2) \\ m_2 \ddot{x}_2 - m_1 \ddot{x}_1 = -2k(x_2 - x_1 - L) - b(\dot{x}_2 - \dot{x}_1) \end{cases}$$

Define the relative coord. $x_{rel} \equiv x_2 - x_1 - L$.

Define the center-of-mass coord. $x_{CM} \equiv \dfrac{m_1 x_1 + m_2 x_2}{M}$, where $M = m_1 + m_2$.

Then $x_1 = x_{CM} - \dfrac{m_2}{M}(x_{rel} + L) \longrightarrow \dot{x}_1 = \dot{x}_{CM} - \dfrac{m_2}{M}\dot{x}_{rel}$

$x_2 = x_{CM} + \dfrac{m_1}{M}(x_{rel} + L) \longrightarrow \dot{x}_2 = \dot{x}_{CM} + \dfrac{m_1}{M}\dot{x}_{rel}$

Substituting and putting $\mu \equiv m_1 m_2 / M$ we get:

$$\begin{cases} M\ddot{x}_{CM} = -b\left(2\dot{x}_{CM} - \dfrac{m_2}{M}\dot{x}_{rel} + \dfrac{m_1}{M}\dot{x}_{rel}\right) \\ m_2 \ddot{x}_{CM} + \mu \ddot{x}_{rel} - m_1 \ddot{x}_{CM} + \mu \ddot{x}_{rel} = -2k x_{rel} - b\dot{x}_{rel} \end{cases}$$

Let's move to the LIF of the center of mass, so that $\dot{x}_{CM} = \ddot{x}_{CM} = 0$.
Then we are left with:

$$\mu \ddot{x}_{rel} = -k x_{rel} - \frac{b}{2}\dot{x}_{rel} \qquad (\#1)$$

<u>In the presence of a GW</u> we know that the geodesic deviation eqn. reads

$$\ddot{\xi}^i = -R^i{}_{0j0}\xi^j \qquad (\text{if } |\xi^i| \ll \lambda_{GW})$$

where ξ^i is the separation b/w 2 geodesics. In this problem

$$\xi^i = (x_{rel}, 0, 0) = (L + (x_2 - x_1), 0, 0)$$

and at first order $\xi^i \approx (L, 0, 0)$, since the perturbation from equilibrium $(x_2 - x_1)$ is due to the wave. As we did in the lecture, we can use the R_{i0j0} computed in the TT gauge and use it here in the LIF, so

$$\ddot{\xi}^i = \frac{1}{2}\ddot{h}^{ij}_{TT}\xi_j \qquad (\#2)$$

Let $h^{TT}_{ij} = \begin{pmatrix} h & 0 \\ 0 & -h \end{pmatrix}\cos\omega t$ at $z = 0$, so eqn. (#1), including the force due to the GW from (#2), becomes:

$$\mu \ddot{x}_{rel} = -k x_{rel} - \frac{1}{2} b\, \dot{x}_{rel} + \frac{L}{2} \ddot{h}_{xx} \mu =$$

$$= -k x_{rel} - \frac{1}{2} b\, \dot{x}_{rel} - \frac{\mu}{2} L h \omega^2 \cos\omega t$$

$$\Rightarrow \boxed{\ddot{x}_{rel} + \frac{1}{2}\frac{b}{\mu} \dot{x}_{rel} + \frac{k}{\mu} x_{rel} = -\frac{1}{2} L h \omega^2 \cos\omega t} \quad (*) \checkmark$$

which is the ODE of a damped oscillator driven by an external oscillating force (the GW).

Let $x_{rel} = A\cos(\omega t + \delta)$, so $\dot{x}_{rel} = -A\omega \sin(\omega t + \delta)$

$$\ddot{x}_{rel} = -A\omega^2 \cos(\omega t + \delta)$$

It's easier using complex notation. Let $x_{rel} = \mathrm{Re}\left[A e^{-i(\omega t + \delta)}\right]$

Plug into $(*)$:

$$\left[-\omega^2 A - \frac{i\omega b}{2\mu} A + \frac{k}{\mu} A\right] \stackrel{!}{=} -\frac{1}{2} L h \omega^2 \cancel{e^{-i\omega t}} \left(\cancel{e^{i\omega t}} e^{i\delta}\right)$$

Equate amplitudes; let $\omega_0^2 \equiv \frac{k}{\mu}$, $\gamma \equiv b/2\mu$, so we have:

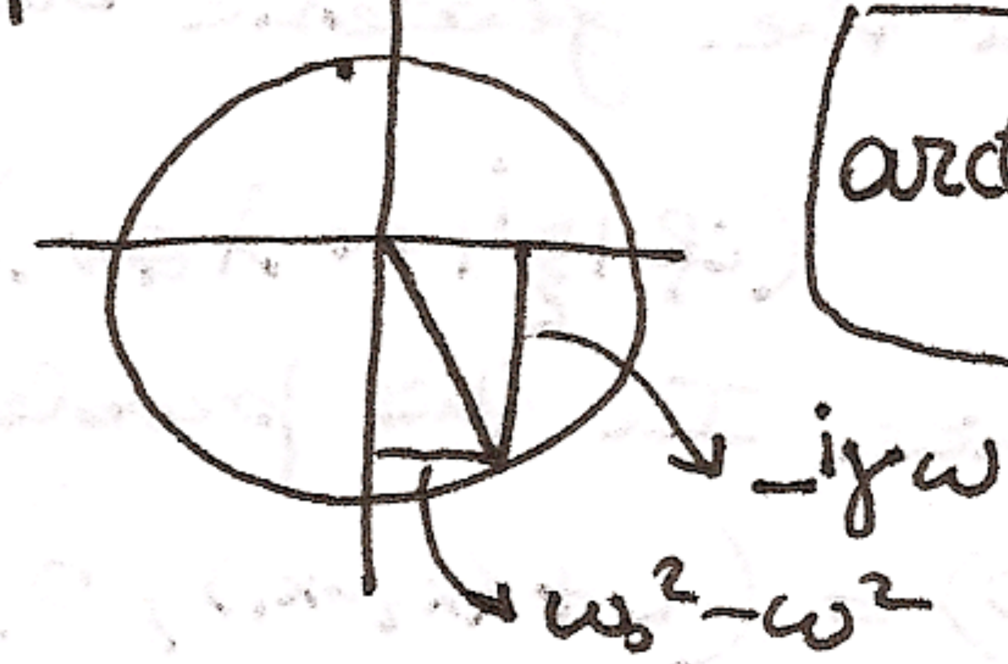
$$\left[\omega_0^2 - \omega^2 - i\gamma\omega\right] A = -\frac{1}{2} L h \omega^2 e^{i\delta}.$$

Take magnitude on both sides: $\sqrt{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}\, |A| \stackrel{!}{=} \frac{L h \omega^2}{2}$

$$\Rightarrow \boxed{|A| = \frac{L h \omega^2/2}{\sqrt{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}}}$$

(#3) there's resonance at $\omega \approx \omega_0$

Equate the phases:



$$\boxed{\arctan\left(-\frac{\gamma\omega}{(\omega_0^2-\omega^2)}\right) \stackrel{!}{=} \delta}$$

At resonance

$$|A| \xrightarrow[\omega \to \omega_0]{} |A|_{max} = \frac{1}{2}\frac{L h \omega_0^{\cancel{2}}}{\gamma \cancel{\omega_0}} = \frac{L h \omega_0}{2\gamma}$$

$$E_{kin} = \frac{1}{2}\mu \dot{x}_{rel}^2 = \frac{1}{2}\mu A^2\omega^2 \sin^2(\omega t+\delta)$$

$$\langle E_{kin}\rangle = \frac{1}{T}\int_0^T E_{kin}\,dt \quad \text{where } T = \frac{2\pi}{\omega}$$

$$= \frac{\omega}{2\pi}\,\frac{1}{2}\mu A^2\omega^2 \underbrace{\int_0^{2\pi/\omega} \sin^2(\omega t+\delta)\,dt}_{\pi/\omega} =$$

$$= \frac{1}{4}\mu A^2\omega^2 \quad \text{where } A \text{ is given in eq. (\#3).}$$ ✓

$$E_{pot} = \frac{1}{2}K x_{rel}^2 = \frac{1}{2}K A^2\cos^2(\omega t+\delta)$$

$$\langle E_{pot}\rangle = \frac{\omega}{2\pi}\,\frac{1}{2}KA^2 \underbrace{\int_0^{2\pi/\omega} \cos^2(\omega t+\delta)\,dt}_{\pi/\omega} = \frac{1}{4}KA^2$$ ✓

$$\langle W\rangle = \frac{1}{T}\int_{1\,cycle} dW = \frac{1}{T}\int_{1\,cycle} F_{GW}\,dx_{rel} = \frac{1}{T}\int_{1\,cycle} F_{GW}\,\dot{x}_{rel}\,dt =$$

$$= \frac{1}{T}\int_0^T\left(-\frac{1}{2}Lh\mu\omega^2\cos\omega t\right)\left(-A\omega\sin(\omega t+\delta)\right)dt =$$

$$= \frac{1}{2T}Lh\mu A\omega^3 \underbrace{\int_0^{2\pi/\omega} \sin(\omega t+\delta)\cos\omega t\,dt}_{\frac{\pi}{\omega}\sin\delta} =$$

$$= \frac{1}{4}Lh\mu A\omega^3\sin\delta \quad \text{where } \delta \text{ is given in (\#4)}$$

The rate of dissipation is: (dissipative force) × (velocity)

$$\frac{dE}{dt} = -\frac{b}{2}\dot{x}_{rel}^2 = -\frac{b}{2}A^2\omega^2\sin^2(\omega t+\delta) = -\mu\gamma A^2\omega^2\sin^2(\omega t+\delta)$$

$$\left\langle\frac{dE}{dt}\right\rangle = -\frac{\omega}{2\pi}\mu\gamma A^2\omega^2 \underbrace{\int_0^{2\pi/\omega}\sin^2(\omega t+\delta)\,dt}_{\pi/\omega} = -\frac{1}{2}\mu\gamma\omega^2A^2$$ ✓

$h = 10^{-21}$

$L = 1\,m$

$\mu = 10^3\,Kg$

$Q = 10^{-6} \equiv \frac{\omega_0}{\gamma}$ (Maggiore 8.21)

$\omega_0 = 10^3\,Hz \times 2\pi$

At resonance $|A|_{max} = \frac{Lh}{2} Q = 5 \times 10^{-28}\ m$

$$\langle E_{kin} \rangle_{res} = \frac{1}{4} \mu \omega_0^2 A_{max}^2 = \frac{1}{\cancel{4}} \cdot 10^3 \cdot \cancel{4}\pi^2 10^6 \cdot 25 \times 10^{-56}\ J =$$

$$= 2.5 \times 10^{-45}\ J$$

$$\langle E_{pot} \rangle_{res} = \frac{1}{4} K A_{max}^2 = \frac{1}{4} \mu \omega_0^2 A_{max}^2 = \frac{1}{4} \cdot 10^3 \cdot 4\pi^2 \cdot 10^6 \cdot 25 \times 10^{-56}\ J$$

$$= \langle E_{kin} \rangle_{res}$$

$$\langle E_{TOT} \rangle_{res} = 2 \langle E_{kin} \rangle_{res} = 5 \times 10^{-45}\ J$$

use $20°C = 293K$

Thermal energy at room temp is $E_{room} \simeq k_B T =$

$$= 1.38 \times 293 \times 10^{-23}\ J$$

$$\simeq 4 \times 10^{-21}\ J$$

$\langle E_{TOT} \rangle_{res} \ll E_{room}$ so the wave cannot be detected because the signal is buried in the thermal noise

(2) In TT gauge $g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \boxed{h^{TT}_{ij}} & & 0 \\ 0 & & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Useful formulas:

Let's omit the label "TT"!

$\boxed{\text{Use } c=1}$

$h^{0\alpha} = 0$

$h^i{}_i = 0$

$\partial^j h_{ij} = 0$

$h^i{}_j = \eta^{ik} h_{kj} = h_{ij}$ at linear order

$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\eta^{\alpha\delta}(\partial_\beta h_{\gamma\delta} + \partial_\gamma h_{\beta\delta} - \partial_\delta h_{\beta\gamma})$ then

$\Gamma^i_{00} = \Gamma^0_{00} = \Gamma^0_{0i} = 0$

$\Gamma^0_{ij} = \frac{1}{2}\partial_0 h_{ij}$

$\Gamma^i_{0j} = \frac{1}{2}\eta^{ik}\partial_0 h_{kj}$, $\Gamma^i_{jk} = \frac{1}{2}\eta^{il}(\partial_j h_{kl} + \partial_k h_{jl} - \partial_l h_{jk})$

$u^\mu = (1,0,0,0)$ so $u^0 = -u_0 = 1$, $u^i = 0$, $u_i = g_{i\alpha}u^\alpha = g_{i0} = 0$

$$\nabla_\mu u_\nu = \cancel{\partial_\mu u_\nu} - \Gamma^\alpha_{\mu\nu} u_\alpha = -\Gamma^0_{\mu\nu} u_0 = \Gamma^0_{\mu\nu}$$

$$\sigma_{\mu\nu} = \nabla_{(\mu} u_{\nu)} + u_{(\mu} u^\alpha \nabla_\alpha u_{\nu)} - \frac{1}{3}(g_{\mu\nu} + u_\mu u_\nu)\nabla_\alpha u^\alpha =$$

$$= \Gamma^0_{\mu\nu} + u_{(\mu} u^0 \nabla_0 u_{\nu)} - \frac{1}{3}(g_{\mu\nu} + u_\mu u_\nu)(\nabla_0 u^0) =$$

$$= \Gamma^0_{\mu\nu} + u_{(\mu} \Gamma^0_{0\nu)} - \frac{1}{3}(g_{\mu\nu} + u_\mu u_\nu)(\Gamma^0_{0\alpha} u^\alpha) =$$

$$= \Gamma^0_{\mu\nu} + u_{(\mu} \Gamma^0_{0\nu)} - \frac{1}{3}(g_{\mu\nu} + u_\mu u_\nu)\cancel{\Gamma^0_{00}} =$$

$$= \Gamma^0_{\mu\nu} - \delta^0_{(\mu} \Gamma^0_{0\nu)}$$

Then

$$\sigma_{00} = \cancel{\Gamma^0_{00}} - \delta^0_0 \cancel{\Gamma^0_{00}} = 0$$

$$\sigma_{0i} = \cancel{\Gamma^0_{0i}} - \delta^0_{(0} \Gamma^0_{0i)} = -\frac{1}{2}\delta^0_0 \cancel{\Gamma^0_{0i}} - \frac{1}{2}\delta^0_i \cancel{\Gamma^0_{00}} = 0$$

$$\sigma_{ij} = \Gamma^0_{ij} - \delta^0_{(i} \Gamma^0_{0j)} = \Gamma^0_{ij} - \frac{1}{2}\cancel{\delta^0_i}\Gamma^0_{0j} - \frac{1}{2}\cancel{\delta^0_j}\Gamma^0_{0i} = \Gamma^0_{ij}$$

$$= \frac{1}{2}\partial_0 h_{ij}$$

The wave eqn for GW in TT gauge is

$$\Box h_{ij}^{TT} = -\frac{16\pi G}{c^4} T_{ij}$$

& using $T_{ij} = -2\eta\sigma_{ij} = -\eta\,\partial_t h_{ij}^{TT}$ we get

$$\boxed{\Box h_{ij}^{TT} = \frac{16\pi G\eta}{c^4}\partial_t h_{ij}^{TT}}$$

Let $h_{ij}^{TT} = A_{ij}\, e^{-i\omega t} e^{ikz}$ plane wave along z. then plugging in:

$$\Box h_{ij}^{TT} = \left(-\frac{1}{c^2}\partial_t^2 + \partial_z^2\right)h_{ij}^{TT} = \left(\frac{\omega^2}{c^2} - k^2\right)h_{ij}^{TT} = \frac{16\pi G\eta}{c^4}(-i\omega)h_{ij}^{TT}$$

$$\Rightarrow \frac{\omega^2}{c^2} - k^2 = \frac{16\pi G\eta}{c^3}(-i\,\omega/c) \underset{\substack{\uparrow\\ \text{definition of } \ell \equiv \left(\frac{8\pi G\eta}{c^3}\right)^{-1}}}{=} -\frac{2i}{\ell}\frac{\omega}{c}$$

$$\boxed{k^2 - \frac{2i\omega}{\ell c} - \frac{\omega^2}{c^2} = 0}$$

$$k^2 = \frac{\omega^2}{c^2} + \frac{2i}{\ell}\frac{\omega}{c}, \qquad c/\omega = \frac{cT}{2\pi} = \lambda\!\!\!^{-}_{GW}$$

$$k = \frac{\omega}{c}\sqrt{1 + 2i\frac{c/\omega}{\ell}} = \frac{\omega}{c}\sqrt{1 + 2i\frac{\lambda\!\!\!^{-}_{GW}}{\ell}}$$

Since $l \propto \frac{c^3}{G}$, it must be a really large number, therefore it is safe to assume that $\tau_{GW} \ll l$:

$$k \approx \frac{\omega}{c}\left(1 + i\frac{\tau_{GW}}{l}\right) = \frac{\omega}{c} + \frac{i}{l}$$

therefore

$$h_{ij}^{TT} = A_{ij}\, e^{-i\omega t}\, e^{i\omega/c\, z}\, e^{-z/l}$$

$$= \underbrace{\left[A_{ij}\, e^{-z/l}\right]}\, e^{-i\omega(t - z/c)}$$

the amplitude decays exponentially along z

$$A_{ij}\frac{1}{e} \stackrel{!}{=} A_{ij}\, e^{-z/l} \Rightarrow e^{-1} = e^{-z/l} \Rightarrow z = l$$ is the distance to travel to get attenuation by $1/e$

$$\eta = 25\, kg/m\cdot s \Rightarrow L = l = \left(\frac{8\pi G\eta}{c^3}\right)^{-1} =$$

$$= \frac{27 \cdot 10^{24}}{8\pi \cdot 6.67 \cdot 10^{-11} \cdot 25}\, m =$$

$$= 6.4 \times 10^{32}\, m \approx 6.8 \times 10^{16}\, ly$$