$$① \quad h = A_0 \frac{G}{c^2}\frac{I_0}{r} + A_1 \frac{G}{c^3}\frac{\dot{I}_1}{r} + A_2 \frac{G}{c^4}\frac{\ddot{I}_2}{r} + \dots +$$

$$+ B_1 \frac{G}{c^4}\frac{\dot{J}_1}{r} + B_2 \frac{G}{c^5}\frac{\ddot{J}_2}{r} \dots =$$

$$= \frac{G}{c^2 r}\sum_{l=0}^{\infty}\frac{A_l}{c^l}\frac{d^l I_l}{dt^l} + \frac{G}{c^4 r}\sum_{l=1}^{\infty}\frac{B_l}{c^{l-1}}\frac{d^l J_l}{dt^l} \qquad (\#\#)$$

where A_l, B_l are real numbers.

$$\left[\frac{G}{c^2 r}\right] = \frac{M\cancel{L}}{\cancel{T}^2}\frac{\cancel{L^2}}{M^{\cancel{2}}}\left(\frac{\cancel{L}}{\cancel{T}}\right)^{-2}\frac{1}{\cancel{L}} = \frac{1}{M}$$

$$\left[\frac{G}{c^4 r}\right] = [c^{-2}]\left[\frac{G}{c^2 r}\right] = \frac{T^2}{L^2 M}$$

$$[I_l] \sim \left[\int x^l \rho\, d^3x\right] = M L^l$$

$$[J_l] \sim \left[\int x^l v \rho\, d^3x\right] = M L^l \frac{L}{T}$$

$$[h] = \left[\frac{G}{r c^2}\right]\sum_{l=0}^{\infty}\left[\frac{A_l}{c^l}\frac{d^l I_l}{dt^l}\right] + \left[\frac{G}{r c^4}\right]\sum_{l=1}^{\infty}\left[\frac{B_l}{c^{l-1}}\frac{d^l J_l}{dt^l}\right] =$$

$$= \underbrace{\frac{1}{\cancel{M}}\sum_{l=0}^{\infty}\frac{\cancel{T^l}}{\cancel{L^l}}\frac{1}{\cancel{T^l}}\cancel{M L^l}}_{\text{dimensionless}} + \underbrace{\frac{T^2}{L^2 \cancel{M}}\sum_{l=1}^{\infty}\frac{\cancel{T^{l-1}}}{\cancel{L^{l-1}}}\frac{1}{T\cancel{^l}}\cancel{M}L^{\cancel{l}}\frac{L}{T}}_{= \frac{\cancel{T^2}}{\cancel{L^2}}\sum_{l=1}^{\infty}\frac{\cancel{L^2}}{\cancel{T^2}} \to \text{dimensionless}}$$

$$\Rightarrow [h] = 1 \quad \text{dimensionless}$$

Consider the l-th term in the expansion; call it h_l, so that $h = \sum_l h_l$. Let's assume a generic term only depends on:

$$\left[\frac{G}{r}\right] = \frac{L^2}{MT^2}$$

$$\left[\frac{d^\alpha}{dt^\alpha}\right] = \frac{1}{T^\alpha}$$

$$[I_\gamma] = ML^\gamma \quad \text{or} \quad [J_\gamma] = \frac{ML^\gamma}{T}$$

$$[c^\beta] = \frac{L^\beta}{T^\beta} \quad , \quad \text{where } \alpha \geq 0 \; , \; \gamma \geq 0 .$$

So the dimensionality of the term is:

1) $$\left[\frac{G}{r}\right]\left[\frac{d^\alpha}{dt^\alpha}\right][I_\gamma][c^\beta] = \frac{L^2}{\cancel{M}T^2}\,\frac{1}{T^\alpha}\,\cancel{M}L^\gamma\,\frac{L^\beta}{T^\beta} = L^{\beta+\gamma+2}\,T^{-(\beta+\alpha+2)}$$

or

2) $$\left[\frac{G}{r}\right]\left[\frac{d^\alpha}{dt^\alpha}\right][J_\gamma][c^\beta] = \frac{L^2}{\cancel{M}T^2}\,\frac{1}{T^\alpha}\,\cancel{M}L^\gamma\,\frac{L}{T}\,\frac{L^\beta}{T^\beta} = L^{\beta+\gamma+3}\,T^{-(\beta+\alpha+3)}$$

Since h is adimensional, each term must be adimensional as well, which implies:

1) $$\begin{cases} \beta+\gamma+2=0 \\ \beta+\alpha+2=0 \end{cases}$$
$$\gamma - \alpha = 0 \Rightarrow \begin{cases} \alpha = \gamma \\ \beta = -\gamma - 2 \end{cases}$$

So the term is: $\dfrac{G}{r}\,\dfrac{d^\alpha}{dt^\alpha}\,I_\alpha\,\dfrac{1}{c^{\alpha+2}}$ with $\alpha \geq 0$

or

2) $$\begin{cases} \beta+\gamma+3=0 \\ \beta+\alpha+3=0 \end{cases}$$
$\alpha = \gamma$ and $\beta = -\gamma - 3$

so the term is: $\dfrac{G}{r}\,\dfrac{d^\alpha}{dt^\alpha}\,J_\alpha\,\dfrac{1}{c^{\alpha+3}}$ with $\alpha \geq 0$

Replace now $\alpha \leftrightarrow l$, then:
$$h = \sum_{l=0}^{\infty}\left[\frac{A_l}{c^2}\,\frac{G}{r}\,\frac{d^l I_l}{dt^l}\,\frac{1}{c^l} + \frac{B_l}{c^3}\,\frac{G}{r}\,\frac{d^l J_l}{dt^l}\,\frac{1}{c^l}\right]$$

with A_l, B_l dimensionless.

This is exactly (##); so we showed that dimensional analysis requires unambiguously form (##) for h.

2) $h \sim \frac{G}{c^4}\frac{\ddot{I}_2}{r} \sim \frac{G}{c^2}\frac{E_{kin}/c^2}{r}$

- <u>Meteorite</u>

$R = 10^3 m \qquad v = 25 Km/s = 25 \times 10^3 m/s$

$\rho \sim 4000 kg/m^3$ (typical density from meteorites.wustl.edu)

$\Rightarrow M = \frac{4}{3}\pi R^3 \rho \cong 1.7 \times 10^{13} kg$

$$I_2 \sim \int \rho(\vec{x}) x^2 d^3x \sim \int M \delta(x - vt)\delta(y)\delta(z) x^2 d^3x$$

$= M v^2 t^2$ for a trajectory $\vec{x}_0(t) = vt\hat{x}$

$\ddot{I}_2 \sim 2Mv^2 \sim E_{kin}$

$\boxed{r \sim \lambdabar} = \frac{c}{2\pi f} = \frac{cT}{2\pi}$ we must be at least $\lambdabar$ away from the source to be in the wave-zone

$T \sim \frac{R}{v} = 0.04 s \Rightarrow \lambdabar \sim 2 \times 10^6 m$

$$h \sim \frac{G}{c^2}\frac{E_{kin}/c^2}{\lambdabar} = \frac{6.67 \times 10^{-11}}{9 \times 10^{16}} \frac{1}{2 \times 10^6} \times \frac{2 \times 1.7 \times 10^{13} \times (25 \times 10^3)^2}{9 \times 10^{16}} =$$

$= 8.7 \times 10^{-29} \sim 10^{-30}$ ✓

- <u>Piezoelectric</u>

$f = 10^8 Hz$

$\vec{x}_0(t) = A\cos(2\pi f t)\hat{x}, \quad \dot{\vec{x}}_0(t) = -2\pi f A \sin(2\pi f t)\hat{x}$

$\ddot{I}_2 \sim E_{kin} \simeq \frac{1}{2} M |\dot{\vec{x}}_0(t)|^2 \sim \frac{1}{2} M (2\pi f A)^2$

A large piezoelectric crystal I found advertised online was

$M \sim 10 Kg$ (www.virtualacquisitionshowcase.com/document/1193/brochure)

with motions of the order of a few mm $\Rightarrow A \sim 1 mm = 10^{-3} m$

$\Rightarrow \frac{E_{kin}}{c^2} \sim \frac{1}{2} \times 10 \times 4\pi^2 \times 10^{16} \times 10^{-6} \frac{1}{9 \times 10^{16}} kg = 2 \times 10^{-5} kg$

$\lambdabar = \frac{c T_{GW}}{2\pi}$

$$\ddot{I}_2 = \frac{d^2}{dt^2}\left(M A^2 \cos^2(2\pi f t)\right) = -8A^2 f^2 \pi^2 M \cos(\underbrace{4\pi f}_{2\pi/T_{GW}} t) \Rightarrow$$

$T_{GW} = \frac{2\pi}{4\pi f} \Rightarrow \lambdabar = \frac{c}{\cancel{2\pi}} \frac{\cancel{2\pi}}{4\pi f} \cong 0.2 m$

$$h \sim \frac{G}{c^2} \frac{E_{kin}/c^2}{r} = \frac{6.67 \times 10^{-11}}{9 \times 10^{16}} \quad \frac{2 \times 10^{-5}}{2 \times 10^{-1}} = 7 \times 10^{-32}$$

③ a) From Maggiore (4.224) and previous formulae, we have that for an ellipsoid which is rigidly rotating about its axis of symmetry with freq. ω_{rot}:

$$h_+ = \frac{4G\omega_{rot}^2}{c^4 r}(I_1 - I_2)\ \frac{1+\cos^2 i}{2}\cos(2\omega_{rot} t)$$

$$h_\times = \frac{4G\omega_{rot}^2}{c^4 r}(I_1 - I_2)\cos i\ \sin(2\omega_{rot} t) \qquad (\#1)$$

where i is the angle b/w the line of sight and the rotation axis, I_1 and I_2 are the principal moments of inertia in the plane $\perp$ rotation axis.

The deviation from ellipticity is defined as: $\varepsilon \equiv \frac{I_1 - I_2}{I_3}$.

Note that $\omega_{GW} = 2\omega_{rot}$. Then

$$h_+ = h_0 \frac{1+\cos^2 i}{2}\cos(\omega_{GW} t)$$

$$h_\times = h_0 \cos i\ \sin(\omega_{GW} t)$$

where $h_0 = \frac{4G\omega_{rot}^2}{c^4 r}\varepsilon I_3 \underset{\uparrow}{=} \frac{4G\pi^2 f_{GW}^2}{c^4 r}\varepsilon I_3 \qquad (\#2)$

$\omega_{rot} = \pi f_{GW}$

So $h_{+,\times} \sim h_0 \propto f_{GW}^2$

Let $\varepsilon = 10^{-4}$, $I_3 \sim 10^{45}\,g\,cm^2 = 10^{38}\,kg\,m^2$.

1) $f_{GW} = 10^2\,Hz$

$$h_0 \cong \frac{33\times 10^{-6}\,m}{r}$$

$$\Rightarrow r \sim \frac{33\times 10^{-6}}{10^{-21}}\,m = 3.3\times 10^{15}\,m \cong 1\ \text{parsec}$$ ✓

2) $f_{GW} = 10^3\,Hz$

r is 100 larger than for $f_{GW} = 10^2\,Hz$, since $h_0 \propto f_{GW}^2$

then $r \sim 100$ parsecs ✓

For a more qualitative derivation of (#1) or (#2) see the ~~xxxxx~~ following. For an ellipsoid with principal axes R, R and $R(1-\varepsilon)$ the moment of inertia in the frame where the body is at rest and aligned with its principal axes is:

$$I_{ij} = \begin{pmatrix} \frac{M}{5}[R^2 + R^2(1-\varepsilon)] & 0 & 0 \\ 0 & \frac{M}{5}[R^2+R^2(1-\varepsilon)] & 0 \\ 0 & 0 & \frac{2}{5}MR^2 \end{pmatrix}$$

where $I_{ij} \equiv \int \rho(\vec{x})(r^2 \delta_{ij} - x_i x_j)\, d^3x$.

In the quadrupole formula we have to ~~take~~ this to an inertial frame and do d^2/dt^2. The trace part ($\sim r^2 \delta_{ij}$) is constant in time so we should look at the trace-free part only

$$I_{ij} - \frac{1}{3}\delta_{ij} \mathrm{Tr}(I_{ab}) = \begin{pmatrix} -\varepsilon \frac{MR^2}{15} & 0 & 0 \\ 0 & -\varepsilon \frac{MR^2}{15} & 0 \\ 0 & 0 & \frac{2}{15}\varepsilon MR^2 \end{pmatrix}$$

where we used $\mathrm{Tr}(I_{ab}) = 6\frac{M}{5}R^2 - \frac{2}{5}MR^2\varepsilon$.

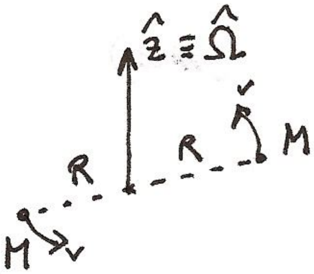
Therefore in the quadrupole formula: $I_2 \sim \varepsilon MR^2$

and: $\ddot{I}_2 \sim \frac{\varepsilon MR^2}{P^2} = \varepsilon MR^2 f_{rot}^2 = \varepsilon MR^2\, 4 f_{GW}^2$

$P \to$ period $P = \frac{1}{f_{rot}}$

hence $h \sim \frac{G}{rc^4} \underbrace{\varepsilon MR^2}_{\sim \text{moment of inertia of sphere}} f_{GW}^2$ which is off by numerical factors from (#2).

b)



Equal-mass binary

$$\frac{\cancel{M}v^2}{R} = \frac{GM^{\cancel{2}}}{(2R)^2} \to v^2 = \frac{GM}{4R} = \Omega^2 R^2$$

$$\to \Omega^2 = \frac{GM}{4R^3} = \frac{2GM}{(2R)^3}$$

So if we put $d \equiv 2R$ and $M_{TOT} = 2M$, we get:

$$\Omega^2 = \frac{GM_{TOT}}{d^3} \qquad (\#3)$$

The trajectory of each object is:

$$① \begin{cases} x_1(t) = R\cos(\Omega t) \\ y_1(t) = R\sin(\Omega t) \\ z_1(t) = 0 \end{cases} \qquad ② \begin{cases} x_2(t) = -x_1(t) \\ y_2(t) = -y_1(t) \\ z_2(t) = 0 \end{cases}$$

$$I_{xx} = \int d^3x\, x^2\, M\left[\delta(\vec{x} - \vec{x}_1(t)) + \delta(\vec{x} - \vec{x}_2(t))\right] = \quad \text{integrand even under } x \to -x$$

$$= 2M \int d^3x\, x^2\, \delta(x - x_1(t))\, \delta(y - y_1(t))\, \delta(z - z_1(t)) =$$

$$= 2MR^2\cos^2(\Omega t)$$

$$I_{yy} = 2MR^2 \sin^2(\Omega t)$$

$$I_{xz} = I_{yz} = I_{zz} = 0$$

$$I_{xy} = 2MR^2 \sin(\Omega t)\cos(\Omega t) = MR^2 \sin(2\Omega t)$$

$$h \sim \frac{G}{c^4 r} \ddot{I}_2 \sim \frac{G}{c^4 r} \ddot{I}_{ij}$$

$$\left.\begin{aligned} \ddot{I}_{xx} &= -4MR^2\Omega^2 \cos(2\Omega t) \\ \ddot{I}_{yy} &= 4MR^2\Omega^2 \cos(2\Omega t) \\ \ddot{I}_{xy} &= -4MR^2\Omega^2 \sin(2\Omega t) \end{aligned}\right\} \Rightarrow 2\Omega = \omega_{GW} = 2\pi f_{GW} \Rightarrow \Omega = \pi f_{GW}$$

then $h \sim \frac{G}{c^4 r} 4MR^2\Omega^2 = \frac{4GM\Omega^2}{c^4 r}\left(\frac{GM}{4\Omega^2}\right)^{2/3} = \frac{2^{2/3}(GM)^{5/3}\Omega^{2/3}}{c^4 r} =$

(#3): $R^3 = \frac{GM}{4\Omega^2}$

$$= \frac{(2\pi)^{2/3}(GM)^{5/3}}{c^4 r} f_{GW}^{2/3} \qquad (\#4)$$

$\Rightarrow h \propto M^{5/3} f_{GW}^{2/3}$

This is correct, by comparison with Maggiore (4.3)

$$h_{+,\times} \sim \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{GW}}{c}\right)^{2/3} \quad \text{with } M_c = \frac{(M_1 M_2)^{3/5}}{(M_1+M_2)^{1/5}} \overset{M_1=M_2=M}{=} \frac{M^{6/5}}{2^{1/5}M^{1/5}} = \frac{M}{2^{1/5}}$$

$$= \frac{4}{r}\frac{G^{5/3}\pi^{2/3}(M/2^{1/5})^{5/3} f_{GW}^{2/3}}{c^{10/3}c^{2/3}} =$$

$$= \frac{4}{r}\frac{G^{5/3}\pi^{2/3}M^{5/3}f_{GW}^{2/3}}{2^{1/3}c^4} =$$

$$= 2\,\frac{(2\pi)^{2/3}(GM)^{5/3}}{c^4 r} f_{GW}^{2/3}$$

note that in our (#4) we are off by a factor of 2 w.r.t. Maggiore since we didn't include that factor in $h \sim \frac{2G\ddot{I}_2}{c^4 r}$

Let $f_{GW} = 100\,\text{Hz}$.

If $M = 1.4 M_\odot$ then $h \sim 110\left(\frac{1\,\text{m}}{r}\right) \Rightarrow$ to have $10^{-21} = h$ we need $r \sim 10^{23}\,\text{m} \cong$
$\cong 3.5\,\text{Mpc}$ ✓

If $M = 10 M_\odot$ then $h \sim 3\times 10^3\left(\frac{1\,\text{m}}{r}\right) \Rightarrow r \sim 3\times 10^{24}\,\text{m} \sim 90\,\text{Mpc}$ ✓

Let $f_{GW} = 10^{-3}\,\text{Hz}$ and $M = 10^6 M_\odot$, then $h \sim 30\times 10^7\left(\frac{1\,\text{m}}{r}\right)$

$\Rightarrow r \sim 30\times 10^{28}\,\text{m} \cong 9400\,\text{Gpc}$ ✓