

# ANDREA TARACCHINI - Week 7

$$\bullet L = L_0 + \frac{1}{c^2} L_2$$

$$L_0 = \frac{m_1}{2} \vec{v}_1^2 + \frac{m_2}{2} \vec{v}_2^2 + \frac{Gm_1 m_2}{r}$$

$$L_2 = \frac{m_1 (\vec{v}_1^2)^2}{8} + \frac{m_2 (\vec{v}_2^2)^2}{8} + \frac{Gm_1 m_2}{2r} \left[ 3(\vec{v}_1^2 + \vec{v}_2^2) - 7\vec{v}_1 \cdot \vec{v}_2 - (\vec{v}_1 \cdot \hat{r})(\vec{v}_2 \cdot \hat{r}) - \frac{G(m_1 + m_2)}{r} \right]$$

$$\vec{p}_1 = \frac{\partial L}{\partial \vec{v}_1} = m_1 \vec{v}_1 + \frac{1}{c^2} \left\{ \frac{m_1}{2} (\vec{v}_1^2) \vec{v}_1 + \frac{Gm_1 m_2}{2r} [6\vec{v}_1 - 7\vec{v}_2 - \hat{r}(\vec{v}_2 \cdot \hat{r})] \right\}$$

$$\vec{p}_2 = \frac{\partial L}{\partial \vec{v}_2} = m_2 \vec{v}_2 + \frac{1}{c^2} \left\{ \frac{m_2}{2} (\vec{v}_2^2) \vec{v}_2 + \frac{Gm_1 m_2}{2r} [6\vec{v}_2 - 7\vec{v}_1 - \hat{r}(\vec{v}_1 \cdot \hat{r})] \right\}$$

$$\text{Let } \vec{\pi}_1 = \frac{1}{2}(\vec{R} - \vec{e}), \quad \vec{\pi}_2 = \frac{1}{2}(\vec{R} + \vec{e}), \quad \vec{p}_1 = \vec{P} - \vec{\beta}, \quad \vec{p}_2 = \vec{P} + \vec{\beta}$$

$$\frac{d\vec{P}}{dt} = \frac{d}{dt}(\vec{p}_1 + \vec{p}_2) \stackrel{\text{Euler-Lagrange eqs.}}{=} \frac{\partial L}{\partial \vec{\pi}_1} + \frac{\partial L}{\partial \vec{\pi}_2} = \frac{\partial L}{\partial R^i} \frac{\partial R^i}{\partial \vec{\pi}_1} + \frac{\partial L}{\partial R^i} \frac{\partial R^i}{\partial \vec{\pi}_2} + \frac{\partial L}{\partial r^i} \frac{\partial r^i}{\partial \vec{\pi}_1} + \frac{\partial L}{\partial r^i} \frac{\partial r^i}{\partial \vec{\pi}_2} = 2 \frac{\partial L}{\partial \vec{R}}$$

Note that  $L = L(r, \vec{\pi}_1, \vec{\pi}_2)$ , then it does NOT depend on  $\vec{R}$ :

$$\frac{\partial L}{\partial \vec{R}} = 0 \Rightarrow \frac{d\vec{P}}{dt} = 0 \Rightarrow \vec{P} \text{ is conserved } \checkmark$$

$$\bullet H = H_0 + \frac{1}{c^2} H_2 = \vec{p}_1 \cdot \vec{\pi}_1 + \vec{p}_2 \cdot \vec{\pi}_2 - L$$

$$\text{At OPN order we have } \vec{p}_1 = m_1 \vec{v}_1 + \mathcal{O}(c^{-2})$$

$$\vec{p}_2 = m_2 \vec{v}_2 + \mathcal{O}(c^{-2})$$

then

$$H_0 = \vec{p}_1 \cdot \frac{\vec{p}_1}{m_1} + \vec{p}_2 \cdot \frac{\vec{p}_2}{m_2} - L_0 = \frac{\vec{p}_1^2}{2m_1} + \frac{\vec{p}_2^2}{2m_2} - \frac{Gm_1 m_2}{r}$$

At 1PN order I can replace  $\vec{v}_{1,2}$  with  $\vec{p}_{1,2}/m_{1,2}$  inside  $L_2$ :

$$L_2 = \frac{1}{8} \frac{(\vec{p}_1^2)^2}{m_1^3} + \frac{1}{8} \frac{(\vec{p}_2^2)^2}{m_2^3} + \frac{Gm_1 m_2}{2r} \left[ 3 \left( \frac{\vec{p}_1^2}{m_1^2} + \frac{\vec{p}_2^2}{m_2^2} \right) - \frac{7\vec{p}_1 \cdot \vec{p}_2}{m_1 m_2} - \frac{(\vec{p}_1 \cdot \hat{r})(\vec{p}_2 \cdot \hat{r})}{m_1 m_2} - \frac{G(m_1 + m_2)}{r} \right]$$

$$H_2 = -L_2 \text{ at 1PN}$$

Move to a frame where  $\vec{P} = 0 \Rightarrow \vec{p}_1 = -\vec{p}_2 = -\vec{p}$ . Use  $\mu$  and  $M$ :

$$H_0 = \frac{\vec{p}^2}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) - \frac{G\mu M}{r} = \frac{\vec{p}^2}{2} \frac{m_1 + m_2}{\mu m_2} - \frac{G\mu M}{r} = \boxed{\frac{\vec{p}^2}{2\mu} - \frac{G\mu M}{r}}$$

$$H_2 = -\frac{1}{8} (\vec{p}^2)^2 \left( \frac{1}{m_1^3} + \frac{1}{m_2^3} \right) - \frac{G\mu M}{2r} \left[ \vec{p}^2 \left( \frac{3}{m_1^2} + \frac{3}{m_2^2} + \frac{7}{m_1 m_2} \right) + \frac{(\vec{p} \cdot \hat{r})^2}{\mu M} - \frac{GM}{r} \right]$$

$$= \frac{m_1^3 + m_2^3}{(\mu M)^3} = \frac{M(m_1^2 + m_2^2 - m_1 m_2)}{(\mu M)^3} = \frac{3(m_1^2 + m_2^2)}{(\mu M)^2} + \frac{7}{\mu M} = \frac{3(M^2 - 2\mu M) + 7\mu M}{(\mu M)^2} = \frac{1}{\mu^2} \left[ 3(1 - 2\frac{\mu}{M}) + \frac{7\mu}{M} \right] = \frac{1}{\mu^2} (3 + \nu)$$

$$= \frac{1}{\mu^3} (M^2 - 3\mu M) = \frac{1}{\mu^3} (1 - 3\frac{\mu}{M}) = \frac{1}{\mu^3} (1 - 3\nu)$$

Then

$$H_2 = \frac{1}{8\mu^3} (3\nu - 1) (\vec{p}^2)^2 - \frac{G\mu M}{2r} \left[ (3 + \nu) \left( \frac{\vec{p}}{\mu} \right)^2 + \frac{(\vec{p} \cdot \hat{r})^2}{\mu M} - \frac{GM}{r} \right] \quad \checkmark$$

We can get rid of  $\mu$  completely:  $\mu = \nu M$ .

• For a circular orbit:

$$\dot{r} = 0, \quad \vec{p} \cdot \hat{r} = \frac{d}{dt} (\vec{p} \cdot \hat{r}) = 0 \Rightarrow \vec{p}_{\text{circ}} = p_{\phi} \frac{\text{circ}}{r} \hat{\phi} \quad (\text{where I introduced polar coords } \{r, \phi\} \text{ in the orbital plane})$$

Note that the angular momentum is

$$\vec{L} = \vec{r} \times \vec{p} = (r \hat{r}) \times \left( \frac{p_{\phi}}{r} \hat{\phi} \right) = p_{\phi} (\hat{r} \times \hat{\phi}) = p_{\phi} \hat{L} \quad \text{where } \hat{L} \text{ is } \perp \text{ orbital plane}$$

Then  $\vec{p}_{\text{circ}}^2 = \frac{L_{\text{circ}}^2}{r^2}$ . In general  $\vec{p}^2 = p_r^2 + \frac{L^2}{r^2}$ , where  $p_r \equiv \vec{p} \cdot \hat{r}$ .

$$E \equiv H|_{\text{circ}} = \frac{L_{\text{circ}}^2}{2\mu r^2} - \frac{G\mu M}{r} + \frac{1}{c^2} \left\{ \frac{(3\nu - 1)}{8\mu^3} \frac{L_{\text{circ}}^4}{r^4} - \frac{G\mu M}{2r} \left[ \frac{(3 + \nu)}{\mu^2} \frac{L_{\text{circ}}^2}{r^2} - \frac{GM}{r} \right] \right\}$$

Note that:  $\dot{r} = \frac{\partial H}{\partial p_r}$  is automatically 0 if  $p_r = 0$ . Therefore the

only eqn. left to exploit is:  $\dot{p}_r = -\frac{\partial H}{\partial r} \stackrel{!}{=} 0$ .

$$\left. \frac{\partial H_0}{\partial r} \right|_{p_r=0} = \frac{G\mu M}{r^2} - \frac{1}{2\mu} \left( \frac{+2L^2}{r^3} \right)$$

(note that this gives us the correct Newtonian result for  $L_{\text{circ}}$ , that is  $G\mu^2 M r$ , so that  $E = -\frac{1}{2} \frac{G\mu M}{r}$ )

$$\left. \frac{\partial H_2}{\partial r} \right|_{p_r=0} = \frac{3\nu - 1}{8\mu^3} 2 \left( \frac{L^2}{r^2} \right) \left( \frac{-2L^2}{r^3} \right) + \frac{G\mu M}{2r^2} \left[ \frac{3 + \nu}{\mu^2} \frac{L^2}{r^2} - \frac{GM}{r} \right] - \frac{G\mu M}{2r} \left[ \frac{3 + \nu}{\mu^2} \left( \frac{-2L^2}{r^3} \right) + \frac{GM}{r^2} \right]$$

$$= -\frac{3\nu - 1}{2\mu^3} \frac{L^4}{r^5} + \frac{G\mu M}{2r^3} \left[ \frac{3 + \nu}{\mu^2} \frac{L^2}{r} - GM + \frac{3 + \nu}{\mu^2} \frac{2L^2}{r} - GM \right]$$

$$= -\frac{3\nu - 1}{2\mu^3} \frac{L^4}{r^5} + \frac{G\mu M}{2r^3} \left[ \frac{9 + 3\nu}{\mu^2} \frac{L^2}{r} - 2GM \right]$$

$$L_{\text{circ}} = L_0 + \frac{1}{c^2} L_2 \quad \text{where } L_0 = \mu \sqrt{GM r}$$

$$\left( \frac{\partial H_0}{\partial r} + \frac{1}{c^2} \frac{\partial H_2}{\partial r} \right) \Big|_{p_r=0} = 0 \quad \text{at IPN I can replace } L \rightarrow L_0 \text{ in } \frac{\partial H_2}{\partial r}$$

and get:

$$\begin{aligned} \frac{\partial H_2}{\partial r} \Big|_{p_r=0} &= -\frac{3\nu-1}{2r^3} \frac{\mu^4 G^2 M^2}{r^3} + \frac{GM}{2r^3} \left[ \frac{9+3\nu}{r^2} \frac{\mu^2 GM r}{r} - 2GM \right] = \\ &= \frac{4\mu G^2 M^2}{r^3} \end{aligned}$$

Then

$$\begin{aligned} \frac{GM}{r^2} - \frac{1}{\mu r^3} (L_0 + \frac{1}{c^2} L_2)^2 + \frac{1}{c^2} \frac{4\mu G^2 M^2}{r^3} &= \\ \approx \underbrace{\left( \frac{GM}{r^2} - \frac{L_0^2}{\mu r^3} \right)}_0 + \frac{1}{c^2} \left( -\frac{2L_0 L_2}{\mu r^3} + \frac{4\mu G^2 M^2}{r^3} \right) &\stackrel{!}{=} 0 \end{aligned}$$

$$\Rightarrow L_2 = \frac{2GM^2 \mu}{\sqrt{GM r}} \quad , \quad \text{at IPN we have:}$$

$$\Rightarrow L_{\text{circ}} = \mu \sqrt{GM r} + \frac{1}{c^2} \frac{2GM^2 \mu}{\sqrt{GM r}} =$$

$$= \mu \sqrt{GM r} \left[ 1 + \frac{1}{c^2} \frac{2GM^2}{GM r} \right] = \boxed{\mu \sqrt{GM r} \left[ 1 + \frac{1}{c^2} \frac{2GM}{r} \right]} \quad \checkmark$$

$$\omega = \frac{\partial H}{\partial L} \Big|_{\substack{L=L_{\text{circ}} \\ p_r=0}} \quad \text{is the orb. freq. for circ. orbits}$$

$$\frac{\partial H_0}{\partial L} \Big|_{\substack{L=L_{\text{circ}} \\ p_r=0}} = \frac{L_0 + \frac{1}{c^2} L_2}{\mu r^2}$$

$$\frac{\partial H_2}{\partial L} \Big|_{\substack{L=L_0 \\ p_r=0}} = -\frac{GM(3+\nu)L_0}{r^3 \mu} + \frac{(3\nu-1)L_0^3}{2r^4 \mu^3} \quad \text{where I neglected the } L_2 \text{ piece at IPN}$$

Substituting the explicit expression for  $L_0, L_2$  and simplifying:

$$\omega = \frac{\sqrt{GM r}}{2r^3} \left[ 2r + \frac{1}{c^2} GM(\nu-3) \right] \Rightarrow \boxed{\omega = \frac{\sqrt{GM}}{r^{3/2}} \left[ 1 + \frac{1}{c^2} \frac{GM(\nu-3)}{2r} \right]} \quad \checkmark$$

Squaring it

$$\omega^2 = \frac{GM}{r^3} \left[ 1 + \frac{1}{c^2} \frac{GM}{r} (v-3) \right]$$

$$v^2 \equiv (GM\omega)^{2/3} = \frac{(GM)^{2/3} (GM)^{1/3}}{r} \left[ 1 + \frac{1}{c^2} \frac{GM}{r} \left( \frac{v}{3} - 1 \right) \right]$$

$$= \frac{GM}{r} \left[ 1 + \frac{1}{c^2} \frac{GM}{r} \left( \frac{v}{3} - 1 \right) \right]$$

$$r = r_0 + \frac{1}{c^2} r_2 \quad \text{where} \quad r_0 = \frac{GM}{v^2}$$

$$\frac{GM}{r_0} \left( 1 - \frac{1}{c^2} \frac{r_2}{r_0} \right) \left[ 1 + \frac{1}{c^2} \frac{GM}{r_0} \left( \frac{v}{3} - 1 \right) \right] = v^2$$

$$\frac{GM}{r_0} + \frac{1}{c^2} \left[ \frac{GM}{r_0} \left( \frac{v}{3} - 1 \right) - \frac{r_2}{r_0} \right] = v^2 \Rightarrow \frac{GM}{r_0} \left( \frac{v}{3} - 1 \right) - \frac{r_2}{r_0} = 0$$

$$\Rightarrow r_2 = GM \left( \frac{v}{3} - 1 \right)$$

$$\Rightarrow r = \frac{GM}{v^2} \left[ 1 + \left( \frac{v}{c} \right)^2 \left( \frac{v}{3} - 1 \right) \right]$$

$$L_{\text{circ}} = \mu \sqrt{GM} \frac{\sqrt{GM}}{v} \left[ 1 + \left( \frac{v}{c} \right)^2 \frac{1}{2} \left( \frac{v}{3} - 1 \right) \right] \left[ 1 + \frac{1}{c^2} \frac{2GM}{GM} v^2 \right]$$

$$= \frac{\mu MG}{v} \left[ 1 + \left( \frac{v}{c} \right)^2 \left( \frac{v}{6} - \frac{1}{2} + 2 \right) \right] =$$

$$= \frac{\mu MG}{v} \left[ 1 + \left( \frac{v}{c} \right)^2 \left( \frac{v}{6} + \frac{3}{2} \right) \right] \quad \checkmark$$

$$E_{\text{circ}} = \frac{L_0^2 + 2L_0L_2/c^2}{2\mu r^2} - \frac{GM\mu}{r} + \frac{1}{c^2} \left\{ \frac{(3v-1)}{8\mu^3} \frac{L_0^4}{r^4} - \frac{GM\mu}{2r} \left[ \frac{(3+v)}{\mu^2} \frac{L_0^2}{r^2} - \frac{GM}{r} \right] \right\}$$

= substitute  $L_0, L_2$  and simplify

$$= -\frac{GM\mu}{8r^2} \left[ +4r + \frac{GM}{c^2} (v-7) \right] = -\frac{GM\mu}{2r} \left[ 1 + \frac{1}{c^2} \frac{GM}{4r} (v-7) \right] =$$

$$= -\frac{GM\mu}{2r_0} \left[ 1 - \frac{1}{c^2} \frac{r_2}{r_0} \right] \left[ 1 + \frac{1}{c^2} \frac{GM}{4r_0} (v-7) \right] =$$

$$= -\frac{\mu v^2}{2} \left[ 1 - \left( \frac{v}{c} \right)^2 \frac{1}{12} (v+9) \right] \quad \checkmark$$

Note:  $x \equiv \frac{v^2}{c^2} = \frac{(GM\omega)^{2/3}}{c^2}$

is dimensionless

Also:  $\frac{GM}{c^2 r}$  is dimensionless

• Start from H (which is already in the frame where  $\vec{P}=0$ )  
 $\mathcal{L}_0 = \frac{1}{2} \mu \vec{v}^2 + \frac{G\mu M}{r}$  (just Newtonian)

$$\mathcal{L}_2 = (-H_2) \Big|_{\vec{P} = \mu \vec{v}} = \frac{1}{8} \mu (1-3\nu) \vec{v}^4 + \frac{G\mu M}{2r} [(3+\nu) \vec{v}^2 + \nu (\hat{r} \cdot \vec{v})^2$$

$$- \frac{GM}{r}] \Big\} \\ \vec{p} = \frac{\partial \mathcal{L}}{\partial \vec{v}} = \mu \vec{v} + \frac{1}{c^2} \left\{ \frac{1}{2} \mu (1-3\nu) \vec{v}^3 + \frac{G\mu M}{r} [(3+\nu) \vec{v} + \nu \hat{r} (\hat{r} \cdot \vec{v})] \right\}$$

$$\frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow E = \text{const.}$$

$$E = \vec{p} \cdot \vec{v} - \mathcal{L} = \mu \vec{v}^2 + \frac{1}{c^2} \left\{ \frac{1}{2} \mu (1-3\nu) \vec{v}^4 + \frac{G\mu M}{r} [(3+\nu) \vec{v}^2 + \nu (\hat{r} \cdot \vec{v})^2] - \frac{1}{2} \mu \vec{v}^2 - \frac{G\mu M}{r} + \frac{1}{c^2} \left\{ -\frac{1}{8} (1-3\nu) \mu \vec{v}^4 - \frac{G\mu M}{2r} [(3+\nu) \vec{v}^2 + \nu (\hat{r} \cdot \vec{v})^2 - \frac{GM}{r}] \right\} \right\} = \\ = \frac{1}{2} \mu \vec{v}^2 - \frac{G\mu M}{r} + \frac{1}{c^2} \left\{ \frac{3}{8} \mu (1-3\nu) \vec{v}^4 + \frac{G\mu M}{2r} [(3+\nu) \vec{v}^2 + \nu (\hat{r} \cdot \vec{v})^2 + \frac{GM}{r}] \right\} \text{ energy for generic orbit } \checkmark$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = 0 \Rightarrow L = \text{const.} = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$$

Use  $\vec{v} = \dot{r} \hat{r} + r \dot{\varphi} \hat{\varphi} \Rightarrow \vec{v}^2 = \dot{r}^2 + r^2 \dot{\varphi}^2$ , then

$$L = \mu r^2 \dot{\varphi} + \frac{1}{c^2} \left\{ \frac{1}{2} \mu (1-3\nu) \vec{v}^2 r^2 \dot{\varphi} + G\mu M (3+\nu) r \dot{\varphi} \right\}$$

Find  $\dot{\varphi}$  to substitute into E. We want to invert  $L = L(r, \dot{r}, \dot{\varphi})$   
 At OPN

$$L = \mu r^2 \dot{\varphi}_0 \Rightarrow \dot{\varphi}_0 = \frac{L}{\mu r^2}$$

At 1PN

$$L = \mu r^2 \dot{\varphi}_0 + \frac{1}{c^2} \left\{ \mu r^2 \dot{\varphi}_2 + \frac{1}{2} \mu (1-3\nu) \vec{v}_0^2 r^2 \dot{\varphi}_0 + G\mu M (3+\nu) r \dot{\varphi}_0 \right\}$$

$$\Rightarrow \dot{\varphi}_2 = -\frac{1}{\mu r^2} \left[ \frac{1}{2} \mu (1-3\nu) \vec{v}_0^2 r^2 \dot{\varphi}_0 + G\mu M (3+\nu) r \dot{\varphi}_0 \right]$$

$$\text{where } \vec{v}_0^2 = \dot{r}^2 + r^2 \dot{\varphi}_0^2$$

Replace  $\dot{\varphi} = \dot{\varphi}_0 + \frac{1}{c^2} \dot{\varphi}_2$  into  $E$ :

$$E = \frac{1}{2} \mu \left[ \dot{r}^2 + r^2 \left( \dot{\varphi}_0 + \frac{1}{c^2} \dot{\varphi}_2 \right)^2 \right] - \frac{GM\mu}{r} + \frac{1}{c^2} \left\{ \frac{3}{8} \mu (1-3\nu) (r^2 + r^2 \dot{\varphi}_0^2)^2 + \right. \\ \left. + \frac{GM\mu}{2r} \left[ (3+\nu) (r^2 + r^2 \dot{\varphi}_0^2) + \nu \dot{r}^2 + \frac{GM}{r} \right] \right\} = \\ = \frac{1}{2} \mu (r^2 + r^2 \dot{\varphi}_0^2) + \frac{1}{c^2} \left\{ \mu r^2 \dot{\varphi}_0 \dot{\varphi}_2 + (\text{all the IPN terms above}) \right\} \Rightarrow$$

$$\frac{E}{\mu} = r^4 \frac{1}{c^2} \left( \frac{3}{8} - \frac{9\nu}{8} \right) + r^2 \left[ \frac{1}{2} + \frac{1}{c^2} \left( \frac{3GM}{2r} + \frac{L^2}{4M^2 r^2 \nu^2} - \frac{3L^2}{4M^2 r^2 \nu} + \frac{GM}{r} \right) \right] \\ + \left[ -\frac{GM}{r} + \frac{L^2}{2M^2 r^2 \nu^2} + \frac{1}{c^2} \left( \frac{G^2 M^2}{2r^2} - \frac{GL^2}{2M r^3 \nu^2} (3+\nu) + \frac{L^4}{8M^4 r^4 \nu^4} (3\nu-1) \right) \right]$$

Newtonian

Neglect the term  $r^4$ . Solve for  $r^2$

$$r^2 = 2 \left( \frac{E}{\mu} + \frac{GM}{r} - \frac{L^2}{2M^2 r^2 \nu^2} \right) + \frac{1}{c^2} \left\{ -\frac{2GM}{r} \frac{E}{\mu} (3+2\nu) + \frac{GL^2}{M \nu^2 r^3} (5+6\nu) \right. \\ \left. + \frac{3L^4}{4M^4 \nu^4 r^4} (1-3\nu) + \frac{1}{r^2} \left[ \frac{L^2}{M^2 \nu^2} \frac{E}{\mu} (3\nu-1) - G^2 M^2 (7+4\nu) \right] \right\} \quad (**)$$

Note that we have the following structure:

$$\frac{1}{2} \dot{r}^2 + V(r) = (\text{constants})$$

Therefore radial perturbation will have a freq

$$\Omega_r^2 = \left. \frac{\partial^2 V}{\partial r^2} \right|_{\text{circ}}$$

I use  $-\frac{1}{2}$  of the RHS of eq. (\*\*) as my radial potential.

I don't have to worry that I'm including constants into  $V$ , as they are killed by  $\partial_r^2$

I also replace  $E, L, r$  with the circ. orbit values at IPN:

$$\frac{E}{\mu} \rightsquigarrow -\frac{\nu^2}{2} \left[ 1 - \left( \frac{\nu}{c} \right)^2 \left( \frac{\nu}{12} + \frac{9}{12} \right) \right], \quad r \rightsquigarrow \frac{GM}{\nu^2} \left[ 1 + \left( \frac{\nu}{c} \right)^2 \left( \frac{\nu}{3} - 1 \right) \right] \\ L \rightsquigarrow \frac{2M^2 G}{\nu} \left[ 1 + \left( \frac{\nu}{c} \right)^2 \left( \frac{\nu}{6} + \frac{3}{2} \right) \right]$$

Then at 1PN I get

$$\Omega^2_{\text{r}} = \frac{v^6}{GM^2} \left[ 1 - 6 \frac{v^2}{c^2} \right]$$

$$GM\Omega = v^3 \Rightarrow \Omega^2_{\text{r}} = \Omega^2 \left( 1 - 6 \frac{v^2}{c^2} \right) \Rightarrow \left( \frac{\Omega_{\text{r}}}{\Omega} \right)^2 = 1 - 6 \frac{v^2}{c^2} \checkmark$$

$$\Rightarrow K = \frac{\Omega}{\Omega_{\text{r}}} = 1 + 3 \frac{v^2}{c^2} \checkmark$$

• In polar coords.

$$H_0 = \frac{1}{2\mu} \left( P_R^2 + \frac{P_\varphi^2}{R^2} \right) - \frac{GM}{R}$$

$$H_2 = \frac{1}{8\mu^3} (3\nu-1) \left( P_R^2 + \frac{P_\varphi^2}{R^2} \right) - \frac{GM}{2R} \left[ \frac{3+\nu}{\mu^2} \left( P_R^2 + \frac{P_\varphi^2}{R^2} \right) + \frac{P_R^2}{\mu M} - \frac{GM}{R} \right]$$

let's use for simplicity  $G=1$  and  $\mu=1$  (see 5.8 in gr-qc/0209039)

$$\frac{H}{\mu} = \frac{\vec{p}^2}{2} - \frac{M}{R} + \frac{1}{c^2} \left\{ \frac{3\nu-1}{8} (\vec{p}^2)^2 - \frac{M}{R} \left[ \frac{3+\nu}{2} \vec{p}^2 + \frac{\nu P_R^2}{2} \right] + \frac{M^2}{2R^2} \right\}$$

with  $\vec{p}^2 = P_R^2 + \frac{P_\varphi^2}{R^2}$ , in polar form. Let  $H \equiv H^{ADM}/\mu$ :

$$\dot{R} = \frac{\partial H}{\partial P_R}, \quad \dot{P}_R = -\frac{\partial H}{\partial R}$$

note that  $H = H(R, P_R, P_\varphi)$

$$\Omega = \frac{\partial H}{\partial P_\varphi}, \quad \dot{P}_\varphi = -\frac{\partial H}{\partial \varphi} = 0$$

For a circular orbit  $R=R_0, P_R=0, \frac{\partial H}{\partial R} \Big|_{R=R_0, P_R=0} = 0 \Leftrightarrow \dot{P}_R = 0$

$$\text{Let } R = R_0 + \delta R$$

$$P_R = \delta P_R$$

$$\Omega = \Omega_0 + \delta \Omega$$

$$P_\varphi = P_{\varphi_0} + \delta P_\varphi$$

At linear order:

$$\begin{aligned} \dot{\delta R} &= \frac{\partial H}{\partial P_R}(R_0 + \delta R, \delta P_R, P_{\varphi_0} + \delta P_\varphi) \stackrel{\text{Taylor}}{=} \underbrace{\frac{\partial H}{\partial P_R}(R_0, 0, P_{\varphi_0})}_{\dot{R}|_{\text{circ}} = 0} + \underbrace{\frac{\partial^2 H}{\partial P_R^2}(R_0, 0, P_{\varphi_0})}_{0 \text{ by } (\#)} \delta R + \\ &+ \frac{\partial^2 H}{\partial P_R^2}(R_0, 0, P_{\varphi_0}) \delta P_R + \underbrace{\frac{\partial^2 H}{\partial P_R \partial P_\varphi}(R_0, 0, P_{\varphi_0})}_{\frac{\partial}{\partial P_\varphi} \left( \frac{\partial H}{\partial P_R} \right)_{R=R_0, P_R=0, P_\varphi=P_{\varphi_0}}} \delta P_\varphi \\ &= C_0 \delta P_R \end{aligned}$$

$\hookrightarrow$  this is linear in  $P_R$ , so it vanishes for  $P_R=0$

Where

$$C_0 \equiv \frac{\partial^2 H}{\partial P_R^2}(R_0, 0, P_{\varphi_0}) = 1 + \frac{1}{c^2} \left\{ \frac{3\nu-1}{2} \left( \frac{P_{\varphi_0}}{R_0} \right)^2 - (3+2\nu) \frac{M}{R_0} \right\}$$

$$\dot{\delta P}_\varphi = 0 \text{ since } \dot{P}_\varphi = 0$$

$$\dot{\delta P}_R = -\frac{\partial H}{\partial R}(R_0 + \delta R, \delta P_R, P_{\varphi_0} + \delta P_\varphi) \stackrel{\text{Taylor}}{=} -\frac{\partial H}{\partial R}(R_0, 0, P_{\varphi_0}) - \frac{\partial^2 H}{\partial R^2}(R_0, 0, P_{\varphi_0}) \delta R$$

0 by virtue of (#)



$$- \frac{\partial^2 H}{\partial R \partial P_R} (R_0, 0, P_{40}) \delta P_R - \frac{\partial^2 H}{\partial R \partial P_\phi} (R_0, 0, P_{40}) \delta P_\phi =$$

again, we have a function linear in  $P_R$  which vanishes when  $P_R = 0$

$$= -A_0 \delta R - B_0 \delta P_\phi$$

where

$$A_0 \equiv + \frac{\partial^2 H}{\partial R^2} (R_0, 0, P_{40}) = \frac{3 P_{40}^2}{R_0^4} - \frac{2M}{R_0^3} + \frac{1}{c^2} \left\{ \frac{3M^2}{R_0^4} - 6M(3+\nu) \frac{P_{40}^2}{R_0^5} + \frac{5(3\nu-1)}{2} \frac{P_{40}^4}{R_0^5} \right\}$$

$$B_0 \equiv + \frac{\partial^2 H}{\partial R \partial P_\phi} (R_0, 0, P_{40}) = -2 \frac{P_{40}}{R_0^3} + \frac{1}{c^2} \left\{ 3M(3+\nu) \frac{P_{40}}{R_0^4} - 2(3\nu-1) \frac{P_{40}^3}{R_0^5} \right\}$$

$$\Omega + \delta \Omega = \frac{\partial H}{\partial P_\phi} (R_0 + \delta R, \delta P_R, P_{40} + \delta P_\phi) = \underbrace{\frac{\partial H}{\partial P_\phi} (R_0, 0, P_{40})}_{\Omega_0} + \frac{\partial^2 H}{\partial R \partial P_\phi} (R_0, 0, P_{40}) \delta R +$$

$$+ \frac{\partial^2 H}{\partial P_R \partial P_\phi} (R_0, 0, P_{40}) \delta P_R + \frac{\partial^2 H}{\partial P_\phi^2} (R_0, 0, P_{40}) \delta P_\phi$$

again  $\frac{\partial H}{\partial P_R}$  is linear in  $P_R$

$$\Rightarrow \delta \Omega = B_0 \delta R + D_0 \delta P_\phi$$

where

$$B_0 = (\text{same as above})$$

$$D_0 \equiv \frac{\partial^2 H}{\partial P_\phi^2} (R_0, 0, P_{40}) = \frac{1}{R_0^2} + \frac{1}{c^2} \left\{ -\frac{M}{R_0^3} (3+\nu) + \frac{3}{2} (3\nu-1) \frac{P_{40}^2}{R_0^4} \right\}$$

Look for solution  $\sim e^{i\sigma t}$

$$\delta P_R = \frac{\delta \dot{R}}{C_0} \Rightarrow \delta P_R = \frac{\delta \ddot{R}}{C_0} = -A_0 \delta R - B_0 \delta P_\phi$$

$$\delta \dot{P}_\phi = 0 \Rightarrow \delta P_\phi = \text{const.}, \text{ without loss of generality put } \delta P_\phi = 0$$

Then

$$\delta \ddot{R} = -A_0 C_0 \delta R \quad \text{use } \delta R \sim e^{i\sigma t}$$

$$-\sigma^2 = -A_0 C_0 \Rightarrow \sigma = \pm \sqrt{A_0 C_0}$$

In order to have stability, the perturbation  $\delta R$  must be oscillatory and not divergent as  $t \rightarrow \infty$ ,  $\sigma$  must be real, thus the condition is:

$$\boxed{A_0 C_0 > 0}$$

Compute  $A_0 C_0$  and replace  $R_0 \rightarrow r_{\text{circ}} = \frac{M}{v^2} \left[ 1 + \left(\frac{v}{c}\right)^2 \left(\frac{v}{3} - 1\right) \right]$

and keep only up to 1PN:

$$P_{\text{po}} \rightarrow l_{\text{circ}} = \frac{M}{v} \left[ 1 + \left(\frac{v}{c}\right)^2 \left(\frac{v}{6} + \frac{3}{2}\right) \right]$$

$$A_0 C_0 = \frac{v^6}{M^2} \left[ 1 - 6 \frac{v^2}{c^2} \right] \stackrel{\uparrow}{=} \boxed{\Omega^2 \left( 1 - 6 \frac{v^2}{c^2} \right)} \quad \checkmark$$

$$A_0 C_0 = 0 \Leftrightarrow \frac{v^2}{c^2} = \frac{1}{6} \quad \text{that is } R = 6M \text{ in geom. units,} \\ \text{the Schw. ISCO} \quad M\Omega = v^3$$

From previous results on periastron

$$\Omega_r^2 = \Omega^2 \left( 1 - 6 \frac{v^2}{c^2} \right) = A_0 C_0$$

So at the ISCO also  $\Omega_r = 0$ . This means that a perturbation of the ISCO has  $\infty$  radial period, i.e. it's a divergent perturbation, and means that we plunge.

The ISCO is the last stable circ. orbit. ✓