Homeworks: Week 7

Recommended readings:

1. Sec. 5.5 in Maggiore's book on strong-field sources and the effacement principle. [The section builds on the article by T. Damour in *300 Years of Gravitation*, edited by S. Hawking and W. Israel, Cambridge University Press.]
2. We discussed only briefly the memory and tail effects. The further-reading section in Maggiore's book suggests several papers on this topic, e.g., K.S. Thorne, Phys. Rev. **45** (1992) 520.

Assignment to be turned in at the beginning of the class on Tuesday, March 26 by students registered to the course:

- State what of the above readings you have done.
- Work the exercise below.

Exercise: Central-force problem at 1PN order (30 points)

We have derived in class the 2-body Lagrangian at 1PN order (i.e., the Einstein-Infeld-Hoffman Lagrangian). Starting from the 1PN-Lagrangian in the coordinates $\mathbf{r}_1$, $\mathbf{r}_2$ and velocities $\mathbf{v}_1$, $\mathbf{v}_2$ (see e.g., Eqs. (5.55) and (5.56) in Maggiore's book for $N = 2$ particles with masses m_1 and m_2):

- Derive the momenta $\mathbf{p}_1$ and $\mathbf{p}_2$. Then, introduce the variables $\mathbf{R} = \mathbf{r}_1 + \mathbf{r}_2$, $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, $\mathbf{P} = (\mathbf{p}_1 + \mathbf{p}_2)/2$, and $\mathbf{p} = (\mathbf{p}_2 - \mathbf{p}_1)/2$, and show that $\mathbf{P}$ is conserved.
- Obtain the relative-motion Hamiltonian at 1PN order in the variables $\mathbf{r}$, $\mathbf{p}$, $M = m_1 + m_2$ and $\nu = m_1\, m_2/M^2$. [Hint: in carrying out the calculation here and below keep only terms at 1PN order!]
- Compute the binding energy E and orbital angular momentum L at 1PN order for circular orbits. Express the final result for E and L in terms of the velocity $v \equiv (M\Omega)^{1/3}$, where Ω is the orbital frequency. [Hint: Impose the circular orbit condition and derive the relation between r and Ω. You will find a few new terms at 1PN order beyond the usual Newtonian relation $M/r^3 = \Omega^2$.]
- Compute the periastron advance at 1PN order for nearly circular orbits. [Hint: It is more convenient to employ the relative-motion Lagrangian. Use the conservation of energy and angular momentum to derive the equation for the radial perturbation around a circular orbit and compute the radial frequency Ω_r as function of Ω. The fractional advance of the periastron per radial period is $\Delta\Phi/(2\pi) = K(\Omega) - 1$, where $K(\Omega) = \Omega/\Omega_r$.]
- Study the stability of circular orbits using the 1PN Hamiltonian.

  Consider the polar coordinates (r, ϕ, p_r, p_ϕ) and a perturbation of the circular orbit defined by

$$
\begin{aligned}
p_r &= \delta p_r \,, \\
p_\phi &= p_\phi^0 + \delta p_\phi \,, \\
r &= r_0 + \delta r \,, \\
\Omega &= \Omega_0 + \delta\Omega \,,
\end{aligned}
$$

where r_0, Ω_0 and p_ϕ^0 refer to the unperturbed circular orbit. Write down the Hamilton equations and linearize them around the circular orbit solution. You should find

$$\begin{aligned}
\delta\dot{p}_r &= -A_0\,\delta r - B_0\,\delta p_\phi\,,\\
\delta\dot{p}_\phi &= 0\,,\\
\delta\dot{r} &= C_0\,\delta p_r\,,\\
\delta\Omega &= B_0\,\delta r + D_0\,\delta p_\phi\,,
\end{aligned} \tag{1}$$

where A_0, B_0, C_0 and D_0 depend on the unperturbed orbit. Determine explicitly A_0, B_0, C_0 and D_0.

Look at solutions of Eqs. (1) proportional to $e^{i\sigma t}$ and find the criterion of stability. [Hint: you should find that there exists a combination Σ_0 of A_0, B_0, C_0 and D_0 such that when $\Sigma_0 > 0$ the orbits are stable. The innermost stable circular orbit (ISCO) corresponds to $\Sigma_0 = 0$].

Express Σ_0 as function of $v = (M\Omega)^{1/3}$ and show that for any value of the binary mass ratio the ISCO at 1PN order coincides with the Schwarzschild ISCO. [This is an accident, which does not hold at high PN orders!]

Finally, show that $\Sigma_0 = 0$ coincides with $\Omega_r = 0$. What is the physical meaning of this result?