- $G_{1PN} = (\vec{q}\cdot\vec{p})\left(c_1 p^2 + \frac{c_2}{q}\right)$

$\vec{n}\cdot\vec{p} = \frac{1}{q}(\vec{q}\cdot\vec{p}) = p_r$

$\vec{N}\cdot\vec{P} = P_r$

$\vec{Q} = \vec{q} + \frac{1}{c^2}\frac{\partial G_{1PN}}{\partial \vec{p}} = \vec{q} + \frac{1}{c^2}\left[\vec{q}\left(c_1 p^2 + \frac{c_2}{q}\right) + 2c_1\vec{p}(\vec{q}\cdot\vec{p})\right]$

$\vec{P} = \vec{p} - \frac{1}{c^2}\frac{\partial G_{1PN}}{\partial \vec{q}} = \vec{p} - \frac{1}{c^2}\left[\vec{p}\left(c_1 p^2 + \frac{c_2}{q}\right) - \frac{c_2}{q^2}\frac{\vec{q}}{q}(\vec{q}\cdot\vec{p})\right] =$

$= \vec{p} - \frac{1}{c^2}\left[\vec{p}\left(c_1 p^2 + \frac{c_2}{q}\right) - \frac{c_2}{q}\vec{n}\,p_r\right]$

$Q^2 = q^2 + \frac{2}{c^2}\vec{q}\cdot\frac{\partial G_{1PN}}{\partial \vec{p}} = q^2 + \frac{2}{c^2}\left[q^2\left(c_1 p^2 + \frac{c_2}{q}\right) + 2c_1 q^2 p_r^2\right] =$

$= q^2\left[1 + \frac{2}{c^2}\left(c_1 p^2 + \frac{c_2}{q} + 2c_1 p_r^2\right)\right]$

$\Rightarrow \boxed{Q = q\left[1 + \frac{1}{c^2}\left(c_1 p^2 + \frac{c_2}{q} + 2c_1 p_r^2\right)\right]}$ up to 1PN

$\Rightarrow \frac{1}{Q} = \frac{1}{q}\left[1 - \frac{1}{c^2}\left(c_1 p^2 + \frac{c_2}{q} + 2c_1 p_r^2\right)\right]$

$P^2 = p^2 - \frac{2}{c^2}\vec{p}\cdot\frac{\partial G_{1PN}}{\partial \vec{q}} = \boxed{p^2 - \frac{2}{c^2}\left[p^2\left(c_1 p^2 + \frac{c_2}{q}\right) - \frac{c_2}{q}p_r^2\right]}$ up to 1PN

$\vec{N}\cdot\vec{P} = P_R = \frac{1}{Q}\vec{Q}\cdot\vec{P} = \frac{1}{Q}\left\{\vec{q}\cdot\vec{p} + \frac{1}{c^2}\left[(\vec{q}\cdot\vec{p})\cancel{\left(c_1 p^2 + \frac{c_2}{q}\right)} + 2c_1 p^2(\vec{q}\cdot\vec{p}) - (\vec{q}\cdot\vec{p})\cancel{\left(c_1 p^2 + \frac{c_2}{q}\right)}\right.\right.$

$\left.\left. + \frac{c_2}{q}p_r^2\right]\right\} = \frac{1}{Q}\left\{q\,p_r + \frac{1}{c^2}\left[2c_1 q p^2 p_r + \frac{c_2}{q}p_r^2\right]\right\} =$

$= \frac{1}{\cancel{q}}\left[1 - \frac{1}{c^2}\left(c_1 p^2 + \frac{c_2}{q} + 2c_1 p_r^2\right)\right]\cancel{q}\left\{p_r + \frac{1}{c^2}\left[2c_1 p^2 p_r + \frac{c_2}{q^2}p_r^2\right]\right\} =$

$= p_r + \frac{1}{c^2}\left[2c_1 p^2 p_r + \frac{c_2}{q^2}p_r^2 - c_1 p^2 p_r - \frac{c_2}{q}p_r - 2c_1 p_r^3\right] \Rightarrow$

$\Rightarrow \boxed{P_R = p_r\left[1 + \frac{1}{c^2}\left(c_1 p^2 - \frac{c_2}{q} + \frac{c_2}{q^2}p_r - 2c_1 p_r^2\right)\right]}$ up to 1PN

$A = 1 + \frac{1}{c^2}\frac{a_1}{Q} + \frac{1}{c^4}\frac{a_2}{Q^2} = 1 + \frac{1}{c^2}\frac{a_1}{q} + \frac{1}{c^4}\left\{\frac{a_2}{q^2} - \frac{a_1}{q}\left[c_1 p^2 + \frac{c_2}{q} + 2c_1 p_r^2\right]\right\}$ up to 1PN (remember that in $g_{00}^{eff}$ the A potential is multiplied by $c^2$)

Similarly for the D potential: $D = 1 + \frac{1}{c^2}\frac{d_1}{q}$

The square of the mapping reads:

$\left(\frac{\hat{H}_{eff}}{c^2}\right)^2 = 1 + 2\frac{\hat{H}_{real}}{c^2}\left(1 + \alpha_1\frac{\hat{H}_{real}}{c^2}\right) + \left(\frac{\hat{H}_{real}}{c^2}\right)^2$ up to 1PN (#)

Plugging the expressions for $A, D, P^2$ and $P_R$ into $\left(\hat{H}_{eff}/c^2\right)^2$ and expanding with Mathematica we get:

$$\left(\frac{\hat{H}_{eff}}{c^2}\right)^2 = 1 + \frac{1}{c^2}\left(p^2 + \frac{a_1}{q}\right) + \frac{1}{c^4}\Big[-2c_1 p^2 + (a_2 - a_1 c_2)\frac{1}{q^2} + (a_1 - a_1 c_1 - 2c_2)\frac{p^2}{q}$$
$$+ (a_1 - 2a_1 c_1 + 2c_2 - d_1)\frac{p_r^2}{q}\Big]$$

Expanding the RHS of eq. (#) up to 2PN we get:

$$RHS = 1 + \frac{1}{c^2}\left(p^2 - \frac{2}{q}\right) + \frac{1}{c^4}\Big[\frac{2\alpha_1 + 3\nu}{4}p^2 + (2 + 2\alpha_1)\frac{1}{q^2} + (-4 - 2\alpha_1 - \nu)\frac{p^2}{q}$$
$$- \nu\frac{p_r^2}{q}\Big]$$

Equating terms with the same structure we get:

$$\begin{cases} a_1 = -2 \\ -2c_1 = \dfrac{2\alpha_1 + 3\nu}{4} \\ a_2 - a_1 c_2 = 2 + 2\alpha_1 \\ a_1 - a_1 c_1 - 2c_2 = -4 - 2\alpha_1 - \nu \\ a_1 - 2a_1 c_1 + 2c_2 - d_1 = \nu \end{cases}$$

If we set $a_2 = 0$ and $d_1 = 0$, we get

$$\begin{cases} a_1 = -2 \\ -2c_1 = \dfrac{2\alpha_1 + 3\nu}{4} & (C1) \\ 2c_2 = 2 + 2\alpha_1 & (C2) \\ -2 + 2c_1 - 2c_2 = -4 - 2\alpha_1 - \nu & (C3) \\ -2 + 4c_1 + 2c_2 = \nu & (C4) \end{cases}$$

It's easy to verify that $\alpha_1 = +\nu/2$, $c_1 = -\nu/2$, $c_2 = \frac{2+\nu}{2}$ is a solution

(C1): $-\cancel{2}\left(-\frac{\nu}{\cancel{2}}\right) \overset{?}{=} \frac{1}{4}\left(\cancel{2}\frac{\nu}{\cancel{2}} + 3\nu\right) \leftrightarrow \nu \overset{?}{=} \frac{1}{4}4\nu$ ok!

(C2): $\cancel{2}\frac{2+\nu}{\cancel{2}} \overset{?}{=} 2 + \cancel{2}\frac{\nu}{\cancel{2}} \leftrightarrow 2+\nu \overset{?}{=} 2+\nu$ ok!

(C3): $-2 + \cancel{2}\left(-\frac{\nu}{\cancel{2}}\right) - \cancel{2}\frac{\nu+2}{\cancel{2}} \overset{?}{=} -4 - \cancel{2}\frac{\nu}{\cancel{2}} - \nu \leftrightarrow -4 - 2\nu \overset{?}{=} -4 - 2\nu$ ok!

(C4): $-2 + 4\left(-\frac{\nu}{2}\right) + \cancel{2}\frac{\nu+2}{\cancel{2}} \overset{?}{=} \nu \leftrightarrow \cancel{-2} - 2\nu + \nu + \cancel{2} \overset{?}{=} -\nu$ ok!

• At 2PN the EOB potentials read

$$A(Q) = 1 - \frac{1}{c^2}\frac{2}{Q} + \frac{1}{c^6}\frac{2\nu}{Q^3}, \quad A' = \frac{2}{c^2 Q^2} - \frac{6\nu}{c^6 Q^4}, \quad A'' = -\frac{4}{c^2 Q^3} + \frac{24\nu}{c^6 Q^5}$$

$$D(Q) = 1 + \frac{1}{c^4}\frac{-6\nu}{Q^2}$$

In polar coordinates: $P^2 = P_R^2 + \frac{P_\varphi^2}{R^2}$, $Q \equiv R$

$$\hat{H}_{eff} = \hat{H}_{eff}(R, P_R, P_\varphi) = c^2\sqrt{A(R)\left[1 + \frac{1}{c^2}\left(P_R^2 + \frac{P_\varphi^2}{R^2}\right) + \frac{1}{c^2}\left(\frac{A(R)}{D(R)} - 1\right)P_R^2\right]}$$

The ISCO is a circular orbit, so $\left.\frac{\partial \hat{H}_{eff}}{\partial R}\right|_{P_R=0} = 0$. It's also the last stable orbit, so $\left.\frac{\partial^2 \hat{H}_{eff}}{\partial R^2}\right|_{P_R=0} = 0$.

Explicitly

$$\left.\frac{\partial \hat{H}_{eff}}{\partial R}\right|_{P_R=0} = c^2 \frac{1}{2\hat{H}_{eff}/c^2}\left.\left\{A'\left[1 + \frac{1}{c^2}\frac{P_\varphi^2}{R^2}\right] + \frac{A}{c^2}\left[-\frac{2P_\varphi^2}{R^3}\right]\right\}\right|_{P_R=0} \stackrel{!}{=} 0$$

$$\Longleftrightarrow A'(R) + \frac{1}{c^2}\frac{P_\varphi^2}{R^2}\left(A' - \frac{2A}{R}\right) \stackrel{!}{=} 0$$

$$\Rightarrow P_\varphi^2 = \frac{R^2 c^2 A'}{\frac{2A}{R} - A'} = c^2 R^2 \frac{RA'}{2A - RA'} = c^2 \frac{R^4 - 3\nu R^2}{R^3 - 3R^2 + 5\nu}$$

$$P_\varphi = \boxed{Rc\sqrt{\frac{RA'}{2A - RA'}} \equiv P_\varphi^{circ}}$$

We use Mathematica to take the $2^{nd}$ derivative $\partial^2/\partial R^2$ of $\hat{H}_{eff}$, then set $P_R = 0$ and $P_\varphi$ equal to $P_\varphi^{circ}$, and simplify

$$\left.\frac{\partial^2 \hat{H}_{eff}}{\partial R^2}\right|_{P_R=0, P_\varphi = P_\varphi^{circ}} = c^2 \frac{1}{RA\sqrt{4A - 2RA'}}\left[-2R(A')^2 + A(3A' + RA'')\right] \stackrel{!}{=} 0$$

$$\Longleftrightarrow -2R(A')^2 + A(3A' + RA'') = 0$$ plugging in $A(R)$ and setting $\boxed{c=1}$:

$$\Rightarrow \boxed{R^5 - 6R^4 + 3\nu R^3 + 20R^2\nu - 30\nu^2 = 0}$$

This eq. must be solved numerically in order to get $R_{ISCO}$ for a given $\nu$.

The freq. is given by

$$\hat{\Omega}_{ISCO} = \left.\frac{\partial \hat{H}_{eff}}{\partial P_\varphi}\right|_{P_R=0, P_\varphi = P_\varphi^{circ}, R = R_{ISCO}} = \frac{P_\varphi^{circ}}{R_{ISCO}^2}\sqrt{\frac{A(R_{ISCO})}{1 + \left(P_\varphi^{circ}/R_{ISCO}\right)^2}}$$

$$\hat{\Omega}_{ISCO}^2 = \left(\frac{1}{R^{\cancel{4}}}\frac{\cancel{R^3}A'}{2A - RA'}\frac{A}{1 + \frac{RA'}{2A - RA'}}\right)_{R=R_{ISCO}} = \left(\frac{AA'}{R}\frac{1}{\cancel{2A - RA'}}\frac{\cancel{2A - RA'}}{2A - \cancel{RA'} + \cancel{RA'}}\right)_{R=R_{ISCO}} =$$

$$= \frac{A'(R_{ISCO})}{2R_{ISCO}} = \frac{R_{ISCO}^2 - 3\nu}{R_{ISCO}^5} \Rightarrow \boxed{\hat{\Omega}_{ISCO} = \sqrt{\frac{R_{ISCO}^2 - 3\nu}{R_{ISCO}^5}}}$$

The energy is given by

$$\underbrace{\frac{E_{ISCO}}{\mu}}_{\hat{E}_{ISCO}} = \hat{H}_{eff}(R_{ISCO}, P_R=0, P_\varphi^{circ}(R_{ISCO})) = A(R_{ISCO})\sqrt{\frac{2}{2A(R_{ISCO}) - R_{ISCO}A'(R_{ISCO})}} \Rightarrow$$

$$\boxed{\hat{E}_{ISCO} = \frac{R_{ISCO}^3 - 2R_{ISCO}^2 + 2\nu}{\sqrt{R_{ISCO}^6 - 3R_{ISCO}^5 + 5\nu R_{ISCO}^3}}}$$

We know that in Schwarzschild (setting $G=c=1$)

$$R_{ISCO} = 6M$$

$$\Omega_{ISCO} M = \frac{1}{(R_{ISCO}/M)^{3/2}} = \frac{1}{6^{3/2}} \approx 0.068$$

$$\frac{E_{ISCO}}{\mu} = \frac{1 - 2M/R_{ISCO}}{\sqrt{1 - 3M/R_{ISCO}}} = \frac{2\sqrt{2}}{3} \approx 0.94$$

The EOB formulae automatically incorporate Schwarzschild when $\boxed{\nu \to 0}$ by construction:

- radius of the ISCO: the 5th order equation becomes

$$R^5 - 6R^4 = R^4(R-6) = 0$$
$$\Rightarrow R_{ISCO} = 6$$

(note that we were working with reduced variables, so no factors of M appear!)

- frequency of the ISCO: $\hat{\Omega}_{ISCO} = \sqrt{\frac{1}{R_{ISCO}^3}}$

- energy of the ISCO: $\hat{E}_{ISCO} = \frac{R_{ISCO}^3 - 2R_{ISCO}^2}{\sqrt{R_{ISCO}^6 - 3R_{ISCO}^5}} = \frac{\cancel{R_{ISCO}^3}}{\cancel{R_{ISCO}^3}} \frac{1 - 2/R_{ISCO}}{\sqrt{1 - 3/R_{ISCO}}}$

• Plunge assuming $P_\varphi = \hat{L}_{ISCO}$, $\hat{E} = \hat{E}_{ISCO}$ throughout. Let $P_R = 0$ at the ISCO. Then we impose

$\hat{E}_{ISCO} \overset{!}{=} \hat{H}_{eff}(R, P_R, \hat{L}_{ISCO}) \Rightarrow$ we can solve for $P_R$ and find:

$$\boxed{P_R^{plunge} = -\frac{\sqrt{D(R)}}{A(R)} \sqrt{\hat{E}_{ISCO}^2 - A(R)\left(1 + \frac{\hat{L}^2_{ISCO}}{R^2}\right)} < 0}$$ valid for $R \leq R_{ISCO}$

By definition, when $R = R_{ISCO}$ we automatically get $P_R = 0$, since the ISCO is a circular orbit. The frequency during plunge is

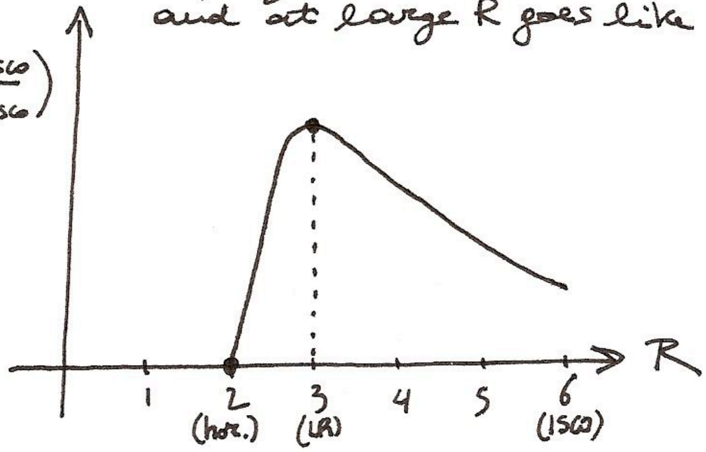
$$\hat{\Omega}_{plunge}(R) = \left.\frac{\partial \hat{H}_{eff}}{\partial P_\varphi}\right|_{\substack{P_R = P_R^{plunge} \\ P_\varphi = \hat{L}_{ISCO}}} =$$

$\hat{\Omega}_{plunge} = 0$ when $A(R) = 0$, that is the horizon of $g^{eff}_{\mu\nu}$: the particle gets stuck on the horizon

$$= \frac{1}{\cancel{2}\hat{E}_{ISCO}} A(R) \frac{\cancel{2}\hat{L}_{ISCO}}{R^2} = \boxed{\frac{\hat{L}_{ISCO}}{\hat{E}_{ISCO}} \frac{A(R)}{R^2}} \propto \frac{1}{R^2}\left(1 - \frac{2}{R} + \frac{2\nu}{R^3}\right)$$

Let's plot for example $\boxed{\nu = 0}$: $\hat{\Omega}_{plunge}$ is 0 on the horizon, peaks at 3 (see below) and at large R goes like $R^{-2}$

$\hat{\Omega}_{plunge} \Big/ \left(\frac{\hat{L}_{ISCO}}{\hat{E}_{ISCO}}\right)$




$\frac{d}{dR}\left[\frac{1}{R} - \frac{2}{R^3}\right] = -\frac{1}{R^2} + \frac{6}{R^4}$

• at $R = 2$ the slope is: $-\frac{1}{4} + \frac{6}{16} \approx .125$

• at $R = 3$, $\frac{1}{R} - \frac{2}{R^3} = \frac{1}{27}$

• at $R = 6$, $\frac{1}{R} - \frac{2}{R^3} = \frac{1}{54}$

$\hat{\Omega}_{plunge}$ peaks at $R_p$ such that

$$\left.\frac{d\hat{\Omega}_{plunge}}{dR}\right|_{R = R_p} = 0 \iff \frac{A'(R_p)R_p^2 - 2R_p A(R_p)}{R_p^4} = 0$$

$$\iff \boxed{A'(R_p)\, R_p = 2A(R_p)}$$

$$\iff \boxed{R_p^3 - 3R_p^2 + 5\nu = 0} \xrightarrow{\nu \to 0} R_p^2(R_p - 3) = 0$$

$$\Downarrow$$

$$\boxed{R_p = 3 = \text{LR in Schw.}}$$

Mathematica gives an analytic solution

$$R_p = 1 + \left[\frac{2}{2 - 5\nu + \sqrt{5\nu(5\nu - 4)}}\right]^{1/3} + \left[\frac{2 - 5\nu + \sqrt{5\nu(5\nu - 4)}}{2}\right]^{1/3}$$