Recommended readings:

1. Sec. 5.5 in Maggiore’s book on strong-field sources and the effacement principle. [The section builds on the article by T. Damour in 300 Years of Gravitation, edited by S. Hawking and W. Israel, Cambridge University Press.]

2. We discussed only briefly the memory and tail effects. The further-reading section in Maggiore’s book suggests several papers on this topic, e.g., K.S. Thorne, Phys. Rev. 45 (1992) 520.

Exercise: Central-force problem at 1PN order (30 points)

Henceforth I use natural units $G = 1 = c$. We have derived in class the 2-body Lagrangian at 1PN order (i.e., the Einstein-Infeld-Hoffman Lagrangian). Starting from the 1PN-Lagrangian in the coordinates $r_1, r_2$ and velocities $v_1, v_2$ (see e.g., Eqs. (5.55) and (5.56) in Maggiore’s book for $N = 2$ particles with masses $m_1$ and $m_2$):

- Derive the momenta $p_1$ and $p_2$. Then, introduce the variables $R = r_1 + r_2$, $r = r_2 - r_1$, $P = (p_1 + p_2)/2$, and $p = (p_2 - p_1)/2$, and show that $P$ is conserved. [10 pts.]

- Obtain the relative-motion Hamiltonian at 1PN order in the variables $r, p, M = m_1 + m_2$ and $\nu = m_1 m_2/M^2$. [Hint: in carrying out the calculation here and below keep only terms at 1PN order! It is also strongly suggested to use Mathematica to manipulate long algebraic expressions.] [10 pts.]

- Compute the binding energy $E$ and orbital angular momentum $L$ at 1PN order for circular orbits. Express the final result for $E$ and $L$ in terms of the velocity $v \equiv (M\Omega)^{1/3}$, where $\Omega$ is the orbital frequency. [Hint: Impose the circular orbit condition and derive the relation between $r$ and $\Omega$. You will find a few new terms at 1PN order beyond the usual Newtonian relation $M/r^3 = \Omega^2$.] [10 pts.]

- Compute the periastron advance at 1PN order for nearly circular orbits. [Hint: It is more convenient to employ the relative-motion Lagrangian. Use the conservation of energy and angular momentum to derive the equation for the radial perturbation around a circular orbit and compute the radial frequency $\Omega_r$ as function of $\Omega$. The fractional advance of the periastron per radial period is $\Delta \Phi/(2\pi) = K(\Omega) - 1$, where $K(\Omega) = \Omega/\Omega_r$.] [optional!]

- Study the stability of circular orbits using the 1PN Hamiltonian. [optional!]

Consider the polar coordinates $(r, \phi, p_r, p_\phi)$ and a perturbation of the circular orbit defined by

\[
\begin{align*}
p_r &= \delta p_r, \\
p_\phi &= p_\phi^0 + \delta p_\phi, \\
r &= r_0 + \delta r, \\
\Omega &= \Omega_0 + \delta \Omega,
\end{align*}
\]

where $r_0$, $\Omega_0$ and $p_\phi^0$ refer to the unperturbed circular orbit. Write down the Hamilton equations and linearize them around the circular orbit solution. You should find

\[
\delta \dot{p}_r = -A_0 \delta r - B_0 \delta p_\phi,
\]
\begin{align*}
\delta \dot{p}_\phi &= 0, \\
\delta \dot{r} &= C_0 \delta p_r, \\
\delta \Omega &= B_0 \delta r + D_0 \delta p_\phi,
\end{align*}
where \( A_0, B_0, C_0 \) and \( D_0 \) depend on the unperturbed orbit. Determine explicitly \( A_0, B_0, C_0 \) and \( D_0 \).

Look at solutions of Eqs. (1) proportional to \( e^{i\sigma t} \) and find the criterion of stability. [Hint: you should find that there exists a combination \( \Sigma_0 \) of \( A_0, B_0, C_0 \) and \( D_0 \) such that when \( \Sigma_0 > 0 \) the orbits are stable. The innermost stable circular orbit (ISCO) corresponds to \( \Sigma_0 = 0 \)].

Express \( \Sigma_0 \) as function of \( v = (M\Omega)^{1/3} \) and show that for any value of the binary mass ratio the ISCO at 1PN order coincides with the Schwarzschild ISCO. [This is an accident, which does not hold at high PN orders!]

Finally, show that \( \Sigma_0 = 0 \) coincides with \( \Omega_r = 0 \). What is the physical meaning of this result?