Each problem is worth 10 points.

1. In an extremely powerful explosion there is a rapid release of energy $E$ in a small region of space, producing an outgoing spherical shock wave whose radius $R$ grows with time $t$. Use dimensional analysis to determine how $R$ depends on $E$, the initial mass density $\rho_0$ of the air, and $t$. Assume that those are the only relevant quantities.

**Solution:** $E \sim ML^2T^{-2}$, $\rho_0 \sim ML^{-3}$, and $t \sim T$. Thus $Et^2 \sim ML^2$, and $Et^2/\rho_0 \sim L^5$. Therefore $R \propto (Et^2/\rho_0)^{1/5}$.

2. The Lennard-Jones potential for the interaction energy between two atoms separated by a distance $r$ takes the form

$$V(r) = \frac{1}{12} r^{-12} - \frac{1}{6} r^{-6}$$

when written in convenient units.

(a) Find the $r$ value $r_{\text{min}}$ at the minimum of the potential.

(b) Find the Taylor expansion of $V(r)$ around $r_{\text{min}}$, keeping terms out through quadratic order in $r - r_{\text{min}}$.

(c) If the motion of a unit mass were governed by this potential, what would be the frequency of its small oscillations around $r_{\text{min}}$?

**Solution:**

$$V(r) = r^{-12}/12 - r^{-6}/6$$
$$V'(r) = -r^{-13} + r^{-7}$$
$$V''(r) = 13r^{-14} - 7r^{-8}$$

(a) $V'(r) = 0$ implies $r^6 = 1$, so $r_{\text{min}} = 1$.

(b) $V(r) = V(1) + V'(1)(r - 1) + (1/2)V''(1)(r - 1)^2 + \ldots = -(1/12) + 3(r - 1)^2 + \ldots$

(c) The potential near the minimum is that of a harmonic oscillator with spring constant $k = V''(1) = 6$. The angular frequency of oscillations for a mass $m = 1$ is $\omega = \sqrt{k/m} = \sqrt{6}$. 
3. The magnetic helicity is a measure of the twisting of magnetic field lines around each other. It is given by an integral over all space, \( \mathcal{H} = \int \mathbf{A} \cdot \mathbf{B} \, dV \), where \( \mathbf{A} \) is the vector potential and \( \mathbf{B} \) is the magnetic field. These vector fields are related by \( \mathbf{B} = \nabla \times \mathbf{A} \).

(a) Show that if \( \mathbf{A} \) is replaced by \( \mathbf{A} + \nabla \lambda \) (a so-called “gauge transformation”), with \( \lambda \) any scalar function, \( \mathbf{B} \) remains unchanged.

(b) Show that the helicity is unchanged under a gauge transformation, assuming the magnetic field goes to zero sufficiently rapidly as the radius grows. (Hint: Integrate by parts using one of the vector calculus product rules, and use the fact that there are no magnetic monopoles.)

FYI: For a perfectly conducting plasma, the helicity is a conserved quantity.

Solution:
(a) The key is that \( \nabla \times \nabla \lambda \) vanishes identically for any (sufficiently differentiable) scalar field \( \lambda \). (This identity holds because mixed partial derivatives commute. For example, the \( x \) component is \( \partial_y \partial_z \lambda - \partial_z \partial_y \lambda \equiv 0 \).) Thus

\[
\mathbf{B}' = \nabla \times (\mathbf{A} + \nabla \lambda) \\
= \nabla \times \mathbf{A} + \nabla \times \nabla \lambda \\
= \mathbf{B}.
\]

(b) The key is the identity \( \nabla \lambda \cdot \mathbf{B} = \nabla \cdot (\lambda \mathbf{B}) - \lambda \nabla \cdot \mathbf{B} \). The absence of magnetic monopoles implies \( \nabla \cdot \mathbf{B} = 0 \), hence \( \nabla \lambda \cdot \mathbf{B} = \nabla \cdot (\lambda \mathbf{B}) \). Thus

\[
\mathcal{H}' = \int (\mathbf{A} + \nabla \lambda) \cdot \mathbf{B} \, dV \\
= \mathcal{H} + \int \nabla \cdot (\lambda \mathbf{B}) \, dV \\
= \mathcal{H} + \int_{\partial V} \lambda \mathbf{B} \cdot d\mathbf{S} \\
= \mathcal{H},
\]

where the divergence theorem is used in the second to last step, and \( \partial V \) is the boundary of the volume, “at infinity”, where \( \mathbf{B} \to 0 \) sufficiently rapidly for the integral to vanish.

4. Let \( f(\theta) \) be the function that is given by 0 for \( -\pi < \theta < 0 \), and by \( \sin \theta \) for \( 0 < \theta < \pi \), and satisfies \( f(\theta + 2\pi) = f(\theta) \). Find all of the non-zero Fourier sine coefficients (don’t worry about the cosine coefficients, even though they are nonzero). (Hints: (i) This is not complicated. (ii) You can relate the integral over \([0, \pi]\) to the one over \([-\pi, \pi]\), and then use a standard identity you proved in a homework problem.)

Solution:

\[
b_n = \pi^{-1} \int_{0}^{\pi} \sin \theta \sin(n\theta) \, d\theta
\]
Thus only the \( n = 1 \) coefficient is nonzero.

5. The temperature \( T(x, t) \) in an infinitely long, thin rod satisfies the heat equation

\[
\partial_t T = \kappa \partial_x^2 T,
\]

where \( \kappa > 0 \) is the heat conductivity. Assume that \( T(x, t) \) may be expressed as a Fourier transform,

\[
T(x, t) = \int \tilde{T}(k, t) e^{ikx} \, dk.
\]  

(a) Insert (12) into the heat equation, and so doing find the differential equation satisfied by the Fourier transform \( \tilde{T}(k, t) \).

(b) Find the solution for \( \tilde{T}(k, t) \) in terms of its initial condition \( \tilde{T}(k, 0) \) at time \( t = 0 \).

(c) Find \( \tilde{T}(k, 0) \) for the case in which the initial temperature distribution is a Dirac delta function, \( T(x, 0) = A\delta(x - a) \), where \( A \) and \( a \) are constants.

FYI: Substituting these results for \( \tilde{T}(k, t) \) in (12), \( T(x, t) \) becomes an explicit integral over \( k \). This yields a Gaussian with center at \( x = a \) and width proportional to \( \sqrt{t} \).

Solution:

(a)

\[
\partial_t T(x, t) = \int \partial_t \tilde{T}(k, t) e^{ikx} \, dk
\]

\[
\kappa \partial_x^2 T(x, t) = \int \kappa (ik)^2 \tilde{T}(k, t) e^{ikx} \, dk
\]

Two functions of \( x \) are equal for all \( x \) if and only if their Fourier transforms are equal for all \( k \), thus \( \partial_t \tilde{T}(k, t) = -\kappa k^2 \tilde{T}(k, t) \).

(b) \( \tilde{T}(k, t) = e^{-\kappa k^2 t} \tilde{T}(k, 0) \).

(c)

\[
\tilde{T}(k, 0) = (1/2\pi) \int T(x, 0) e^{-ikx} \, dx
\]

\[
= (1/2\pi) \int A\delta(x - a) e^{-ikx} \, dx
\]

\[
= (A/2\pi)e^{-ika}.
\]

FYI: This yields \( T(x, t) = (A/2\pi) \int e^{-\kappa k^2 t} e^{i(x-a)k} \, dk = (A/\sqrt{4\pi\kappa t})e^{-(x-a)^2/4\kappa t} \).
6. Evaluate the integral \( \int_{-\infty}^{\infty} e^{-x} \delta(3 + x^{-1}) \, dx \).

**Solution:** If a function \( g(x) \) is zero only at some \( x_0 \), then \( \delta(g(x)) = |g'(x_0)|^{-1} \delta(x-x_0) \). In the present case \( g(x) = 3 + x^{-1} \), so \( x_0 = -1/3 \), and \( |g'(x_0)| = |-x_0^{-2}| = 9 \). Hence the integral is equal to \( e^{1/3}/9 \).

7. Two objects of mass \( m \) lie on a frictionless table, connected to each other with a spring constant \( k \) and connected to opposite walls with spring constant \( k \) for the mass on the left and \( 2k \) for the mass on the right. Consider only motions along a straight line.

(a) Write the coupled equations of motion (Newton's second law) for the displacements \( x_1 \) and \( x_2 \) of the left and right masses from their equilibrium positions.

(b) Find all the normal mode frequencies. How many are there?

(c) Find the ratio \( x_2/x_1 \) of the displacements for the two masses in each of the normal modes. For each mode, state whether the masses move in the same direction or oppositely.

**Solution:**

(a) \[
\begin{align*}
    m\ddot{x}_1 &= -kx_1 - k(x_1 - x_2) = -2kx_1 + kx_2 \\
    m\ddot{x}_2 &= -k(x_2 - x_1) - 2kx_2 = kx_1 - 3kx_2
\end{align*}
\]

(b) In matrix form, the equations are \( \ddot{\mathbf{x}} = A\mathbf{x} \), where

\[
A = \frac{k}{m} \begin{pmatrix}
-2 & 1 \\
1 & -3
\end{pmatrix}.
\]

The eigenvalues of \( A \) are given by the characteristic equation,

\[
\det(A - \lambda I) = (\lambda + 2)(\lambda + 3) - 1 = \lambda^2 + 5\lambda + 5 = 0,
\]

in units with \( k/m = 1 \). The roots are \( \lambda_{\pm} = (-5 \mp \sqrt{5})/2 \). The frequencies are given by \( \omega^2 = -\lambda \), i.e. \( \omega_{\pm} = \sqrt{(5 \pm \sqrt{5})/2\sqrt{k/m}} \).

(c) The eigenvectors are determined by \( (A - \lambda I)\mathbf{x} = 0 \), i.e.

\[
\begin{pmatrix}
-2 - \lambda & 1 \\
1 & -3 - \lambda
\end{pmatrix}
\begin{pmatrix}
e_1 \\
e_2
\end{pmatrix} =
\begin{pmatrix}
-(\lambda + 2)e_1 + e_2 \\
e_1 - (\lambda + 3)e_2
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]

Hence \( e_2 = (\lambda + 2)e_1 \), so \( x_2/x_1 = e_2/e_1 = \lambda + 2 = (-1 \mp \sqrt{5})/2 = \{-1.618, 0.618\} \). The \( \omega_- \) mode has the smaller frequency, and \( x_2/x_1 = 0.618 > 0 \), so the masses move in the same direction, with the second mass moving less. The \( \omega_+ \) mode has the higher frequency, and \( x_2/x_1 = -1.618 \), so the masses move in opposite directions, with the second mass moving more.