Prof. Alessandra Buonanno
Room 4212, (301)405-1440
buonanno@umd.edu

1. Consider a binary system composed of two orbiting black holes each of mass M. The system emits gravitational waves and the black holes spiral inwards until they finally coalesce and become one black hole. The last burst of gravitational radiation is emitted with some maximum power P (energy per unit time). The only quantities P may depend on are M, Newton's constant G, and the speed of light c. (The speed of light enters in Einstein's field equation of general relativity.) Determine the dependence of P on these quantities.

   Solution: To find the dimensions of G think of an equation involving G that you know, e.g. the force law $F = Gm_1m_2/r^2$ between two masses at a distance r. Thus $G = Fr^2/m_1m_2 \sim (MLT^{-2})L^2/M^2 = M^{-1}L^3T^{-2}$, $c \sim LT^{-1}$, and $M \sim M$, hence $G^a c^b M^c \sim M^{c-a}L^{3a+b}T^{-2a-b}$. Since $P \sim ML^2T^{-3}$ we have $c - a = 1$, $3a + b = 2$, and $-2a - b = -3$. Adding the last two equations yields $a = -1$, then the first and second yield $c = 0$ and $b = 5$. Thus $P \propto c^5/G = 3.6 \times 10^{59}$ erg/s, a staggeringly high power. Interestingly, it does not depend on the mass M.

   More insight is provided by considering the "radius" R of a black hole of mass M. For the purpose of dimensional analysis we may think of this in Newtonian terms, as the radius for which the escape velocity $v_{esc} = (2GM/R)^{1/2}$ is equal to the speed of light, i.e. $R = 2GM/c^2$. At the moment of maximum power the black holes are just about to coalesce, and they are moving at a speed near the speed of light. (No other quantity with dimensions of speed enters the problem.) A binding energy $E_B \sim GM^2/R \sim Mc^2$ is thus released in the time $\tau \sim R/c \sim GM/c^3$ it takes something moving at the speed of light to cross the distance R. The corresponding power is $P \sim E_B/\tau \sim c^5/G$. (The actual value has a dimensionless numerical factor.) The mass M drops out since both the energy released and the time over which it is released are proportional to M.

2. Consider the quadratic equation $x^2 + 2ax - 3 = 0$, with a a small positive number. (a) Find the positive root exactly using the quadratic formula. (b) Expand your exact root in a series in powers of a, keeping terms up through $O(a^2)$. (c) Now go back to the beginning and solve for the positive root using the method of series solutions, up through $O(a^2)$. (You should recover the same result as found in part (b)!)

Solution:
(a) $x_+ = -a + \sqrt{a^2+3}$.
(b) $x_+ = -a + \sqrt{3}(1 + a^2/3)^{1/2} = \sqrt{3} - a + (1/2\sqrt{3})a^2 + \ldots$. (Here I used $(1+\epsilon)^n = 1 + n\epsilon + O(\epsilon^2)$, which as you see is very efficient. Alternatively, you could compute the first and second derivatives of $x_+(a)$ with respect to a, evaluated at $a = 0$, and use these to write the first three terms in the Taylor expansion of $x_+(a)$.)
(c) $x = x_0 + x_1 a + x_2 a^2 + \ldots$, hence $x^2 = x_0^2 + 2x_0x_1a + (x_1^2 + 2x_0x_2)a^2 + \ldots$. When $a = 0$ the equation reduces to $x^2 - 3 = 0$, so the positive root is $\sqrt{3}$, so we could put $x_0 = \sqrt{3}$ from the beginning. But let's see how the pattern develops without doing this. Inserting the power series expansions into the equation and keeping terms through $O(a^2)$ we have

$$\begin{aligned} x^2 + 2ax - 3 &= x_0^2 + 2x_0x_1a + (x_1^2 + 2x_0x_2)a^2 + 2a(x_0 + x_1a) - 3 + \ldots \quad (1) \\ &= (x_0^2 - 3) + (2x_0x_1 + 2x_0)a + (x_1^2 + 2x_0x_2 + 2x_1)a^2 + \ldots \quad (2) \end{aligned}$$

Since this is zero for any value of a, the coefficient of each power of a must vanish separately. Working up from the lowest order we have $x_0 = \sqrt{3}$, $x_1 = -1$, $x_2 = -(2x_1 + x_1^2)/2x_0 = 1/2\sqrt{3}$. This agrees with the expansion of the exact solution.

3. A particle of mass m moves in a one dimensional potential that takes the form $V(x) = x^3 - 3x$ in some units. (a) Sketch $V(x)$. Label with their x value the points where $V(x) = 0$, and the points where $V'(x) = 0$. Label the stable and unstable equilibrium points. (b) Find the Taylor series approximation of $V(x)$ expanded around the stable equilibrium point, dropping any terms higher than quadratic order. (c) Find the frequency of small oscillations around the stable equilibrium point. (*Hint*: compare to a harmonic oscillator.)

Solution:

(a) The potential vanishes at $x = 0, \pm\sqrt{3}$. The potential behaves like x^3 as $x \to \pm\infty$, so it goes to $\pm\infty$. The slope at the origin is -3 at $x = 0$. The equilibrium points lie where $V'(x) = 3x^2 - 3 = 0$, i.e. at $x = \pm 1$, and $V(\pm 1) = \mp 2$. It is clear from the position of the zeros and the signs of $V(x)$ at large and small x that $+1$ is a minimum, hence stable, and -1 is a maximum, hence unstable. This can also be seen from the sign of the second derivative: $V''(x) = 6x$, so $V''(\pm 1) = \pm 6$. The second derivative is positive(negative) at a minimum(maximum).

(b) $V(x) = V(1) + V'(1)(x-1) + (1/2)V''(1)(x-1)^2 + \ldots = -2 + 3(x-1)^2 + \ldots$

(c) The angular frequency of an oscillator with mass m and potential $(1/2)kx^2$ is $\omega = \sqrt{k/m}$. (The oscillator satisfies $m\ddot{x} = -kx$, i.e. $\ddot{x} = -(k/m)x$. The general solution is $x(t) = A\sin(\omega t + \delta)$, where $\omega = \sqrt{k/m}$ is the angular frequency.) Comparing $(1/2)V''(1)(x-1)^2$ with $(1/2)kx^2$, we see that the effective spring constant for small oscillations in the potential $V(x)$ around $x = 1$ is given by $k_{eff} = V''(1) = 6$, so $\omega = \sqrt{6/m}$.

4. (a) Let $f(x,y) = x^3 + 2x^2y$, and find ∇f.

   (b) Find the rate of change of f with respect to distance in the direction of the vector $3\hat{\mathbf{x}} + 4\hat{\mathbf{y}}$ at the point $(x,y) = (1,1)$.

   (c) Find the rate of change of f with respect to distance in the direction of maximum *decrease* at the point $(x,y) = (1,1)$.

   (d) What is the angle between the vector $3\hat{\mathbf{x}} + 4\hat{\mathbf{y}}$ and the direction of maximum *decrease* at the point $(x,y) = (1,1)$?

   Solution:

   (a) $\nabla f = (3x^2 + 4xy)\,\hat{\mathbf{x}} + 2x^2\,\hat{\mathbf{y}}$.

   (b) At $(1,1)$ we have $\nabla f = 7\,\hat{\mathbf{x}} + 2\,\hat{\mathbf{y}}$. The unit vector in the direction of the specified vector is $\hat{\mathbf{n}} = (3\hat{\mathbf{x}} + 4\hat{\mathbf{y}})/5$. The rate of change wrt distance in this direction is $\hat{\mathbf{n}} \cdot \nabla f = (7 \cdot 3 + 2 \cdot 4)/5 = 29/5$.

   (c) The direction of maximum decrease is the direction opposite to the direction of ∇f, and the rate of change in this direction is $-|\nabla f| = -\sqrt{49+4} = -\sqrt{53}$.

   (d) $\cos\theta = (-\hat{\mathbf{n}} \cdot \nabla f)/|\nabla f| = -29/5\sqrt{53}$.

5. Let $\mathbf{F} = r^n\,\mathbf{r}$, where r is the radial distance from the origin, $\mathbf{r}$ is the radial position vector, and n is a constant.

   (a) Evaluate $\nabla \cdot \mathbf{F}$. (*Useful check*: The value of n for which your result vanishes should agree with what you know about the electric field of a point charge.)

   (b) Using only your result from part (5a), evaluate the integral $\int_{\mathcal{V}} \nabla \cdot \mathbf{F}\, dV$, where $\mathcal{V}$ is the spherical shell between the two radii R_1 and $R_2 > R_1$.

   (c) Evaluate $\oint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{dS}$, where $\mathcal{S}$ is the surface of a sphere of radius R surrounding the origin.

   (d) Using only Gauss' theorem and your result from part (5c), re-evaluate the integral you computed in part (5b). (You should get the same result, but you must derive it here using Gauss' theorem.)

   Solution:

   (a) x Using the method from the homework, we can apply the product rule for divergence, the chain rule, and the identities $\nabla r = \hat{\mathbf{r}}$ and $\nabla \cdot \mathbf{r} = 3$, viz.
   $\nabla \cdot \mathbf{F} = \nabla \cdot (r^n\,\mathbf{r}) = \nabla(r^n) \cdot \mathbf{r} + r^n \nabla \cdot \mathbf{r} = nr^{n-1}\nabla r \cdot \mathbf{r} + 3r^n = (n+3)r^n$.
   Alternatively, we can use the formula for divergence in spherical coordinates. Since $\mathbf{F} = r^{n+1}\hat{\mathbf{r}}$, we have $\nabla \cdot \mathbf{F} = r^{-2}\partial_r(r^2 F_r) = r^{-2}\partial_r(r^2 r^{n+1}) = (n+3)r^n$. Note that the dimensions match properly.

   (b) $\int_{\mathcal{V}} \nabla \cdot \mathbf{F}\, dV = \int_{R_1}^{R_2} (n+3)r^n\, r^2 dr d\Omega = 4\pi(R_2^{n+3} - R_1^{n+3})$. ($d\Omega = \sin\theta d\theta d\varphi$.)

   (c) $\oint_{\mathcal{S}} \mathbf{F} \cdot \mathbf{dS} = \int r^n \mathbf{r} \cdot \hat{\mathbf{r}} r^2 d\Omega = 4\pi R^{n+3}$.

   (d) $\int_{\mathcal{V}} \nabla \cdot \mathbf{F} = \oint_{\partial\mathcal{V}} \mathbf{F} \cdot \mathbf{dS} = \oint_{R_2} \mathbf{F} \cdot \mathbf{dS} - \oint_{R_1} \mathbf{F} \cdot \mathbf{dS} = 4\pi(R_2^{n+3} - R_1^{n+3})$.