HW#8 —Phys374—Fall 2013 Prof. Alessandra Buonanno
Due before class, Thursday, Nov. 14, 2013 Room 4212, (301)405-1440
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Note: *Be sure to fully justify the relation between the given integral and the closed contour integral you employ in the following problems.*

1. Find the residues of the following functions at the given values of z:

   (a) $(z + z^2)^{-1}$ at 0 and at -1.
   (b) $z^{-2} \ln(1 + 2z)$ at 0
   (c) $[z^3(z + 2)^2]^{-1}$ at 0 and at -2.
   (d) $\cos z/(2z - \pi)^4$ at $\pi/2$
   (e) $(z^2 + 1)^{-3}$ at $\pm i$.

   Hint: See the supplement for methods of evaluating residues. [$3 \times 5 = 15$ pts.]

   Solution:
   (a) $(z + z^2)^{-1} = z^{-1}(1 + z)^{-1}$. This has the form $f(z)/(z - z_0)$ with $f(z)$ non-zero, hence the residue is $f(z_0)$, i.e. 1 at $z = 0$ and -1 at $z = -1$.

   (b) $\ln(1 + 2z) = 2z + O(z^2)$, so the Laurent series is $2/z + O(1)$, so the residue is 2.

   (c) This has the form $f(z)/(z - z_0)^m$, so the residue is $(1/(m - 1)!)f^{(m-1)}(z_0)$. At $z = 0$, the residue is 1/2 the second derivative of $(z + 2)^{-2}$, i.e. 3/16. At $z = -2$ the residue is the derivative of z^{-3}, i.e. $-3/16$.

   (d) Write this as $(2^{-4} \cos z)/(z - \pi/2)^4$, so it has the form of the previous problem. The residue is then 1/3! times the third derivative of the numerator at $\pi/2$, i.e. $(1/3!)(1/16) = 1/96$.

   (e) Write this as $(z + i)^{-3}(z - i)^{-3}$, so it has the standard form again. The residue at $\pm i$ is 1/2 the second derivative of $(z \pm i)^{-3}$, i.e. $(1/2)12(\pm 2i)^{-5} = \mp 3i/16$.

2. (a) Show using contour integration that

$$\int_0^\infty \frac{\cos mx \, dx}{x^2 + a^2} = \frac{\pi}{2a} e^{-ma}$$

   Hint: See the similar example in the textbook.

   (b) Explain in words why the result decays so rapidly (i) as m grows with a fixed, and (ii) as a grows with m fixed. [8+2=10 pts.]

Solution:

a) The integrand is even so we can extend the integral over the entire real axis, dividing by two. You might think the next step would be to regard x as the real part of z, hence to replace x by z in the integrand, and then to write $\cos mz = (e^{imz}+e^{-imz})/2$. This is allowed of course, however what about the behavior at infinity? If $z = R\cos\theta + iR\sin\theta$ then $e^{\pm imz} = e^{\mp mR\sin\theta}e^{\pm imR\cos\theta}$. If $m > 0$ the upper sign case goes to zero in the upper half plane and blows up in the lower half plane, and vice versa for the lower sign. We could handle this by breaking the integral into two parts, closing the contour in the upper half plane for one part and in the lower half plane for the other part. Instead, we can deal with just one integral by noting from the beginning, *before* replacing the real number x by z, that $\cos mx = Re(e^{imx})$. Thus the integral we wish to evaluate is $I = Re[(1/2)\int_{-\infty}^{\infty} dx e^{imx}/(x^2+a^2)] = Re[(1/2)\oint dz e^{imz}/(z^2+a^2)]$, where the contour closes on an arc at infinity in the upper half plane. The poles of the integrand are at $z = \pm ia$, so we enclose the pole at $z = ia$, where the residue is $e^{im(ia)}/(ia+ia) = e^{-ma}/2ia$. Multiplying by $2\pi i$, we find for the integral $I = (\pi/2a)e^{-ma}$.

b) The numerator oscillates symmetrically. If the denominator were strictly constant the cosine would average to zero. As m gets larger with fixed a, the oscillation 'wavelength' gets smaller, so the denominator changes less over one wavelength, i.e. it is closer to being constant. We see from our evaluation that this cancellation of the oscillations drives the integral to zero exponentially as m grows. If instead m is fixed, the 'wavelength' $\lambda = 2\pi/m$ is fixed. As a grows the denominator is more and more relatively constant over a distance λ, so cancellation of the oscillations becomes more and more complete.

3. (a) Evaluate the integral $\int_0^\infty dx/(x^n+1)$, where $n \geq 2$ is a positive integer, by relating it to the contour integral around the boundary of an infinite piece of pie with edges $\theta = 0$ and $\theta = 2\pi/n$, together with the arc at infinity that joins these edges. (b) Show that the result approaches 1 as $n \to \infty$, and explain with reference to the behavior of the integrand why this is the limiting value. (*Answer*: $(\pi/n)/\sin(\pi/n)$.) [8+2=10 pts.]

Solution: **a**) The contribition from the arc at radius R goes as $\sim R/R^n$, hence that from the arc at infinity vanishes. Along the line $\theta = 2\pi/n$ we have $z = re^{i2\pi/n}$, so $z^n = r^n$ and $dz = dr\, e^{i2\pi/n}$. On this part of the contour r runs from ∞ to 0, so if the original integral is I, this contributes $-e^{i2\pi/n}I$. We thus have $(1-e^{i2\pi/n})I = \oint dz/(z^n+1)$. The poles are simple poles at the nth roots of -1, of which only $e^{i\pi/n}$ is enclosed by our contour. The integrand has the form $1/g(z)$ so the residue is $1/g'(e^{i\pi/n}) = 1/ne^{i\pi(n-1)/n} = -1/(ne^{-i\pi/n})$. Multiplying by $2\pi i$ yields $I = (2\pi i)(-1/n)[e^{-i\pi/n}(1-e^{i2\pi/n})]^{-1} = (\pi/n)/\sin(\pi/n)$.

b) As $n \to \infty$ we have $\pi/n \to 0$, so $\sin(\pi/n) \to \pi/n$, hence $(\pi/n)/\sin(\pi/n) \to 1$. Looking at the integrand, as $n \to \infty$ we have $x^n \to 0$ if $|x| < 1$, and $x^n \to \infty$ if $|x| > 1$. Thus the integrand goes to 1 if $|x| < 1$ and to zero if $|x| > 1$. The integral thus reduces to $\int_0^1 dx = 1$.

4. Consider potential flow perpendicular to an infinite solid cylinder of radius R. This reduces to a two-dimensional problem in the xy plane. The cylinder intersects the plane in a disk. For the boundary condition "at infinity", suppose that far from the cylinder in all directions the velocity is $\mathbf{v} = v_0\hat{\mathbf{x}}$. The boundary condition at the cylinder surface is that there is no flow perpendicular to the cylinder, so $\mathbf{v} \cdot \hat{\mathbf{r}} = 0$, taking the origin of polar coordinates at the center of the cylinder. That is, the partial derivative of the velocity potential with respect to radius r (at fixed angle) vanishes at $r = R$. To solve for the flow we need only find an analytic function of $z = x + iy$ whose real part satisfies the appropriate boundary conditions. [2+5+3+2+2+1=15 pts.]

(a) Find an analytic function $h_1(z)$ whose real part is a potential for the velocity field $\mathbf{v} = v_0\hat{\mathbf{x}}$.

Solution: $h_1(z) = v_0 z$.

(b) Find an analytic function $h_2(z)$, to be added to your function $h_1(z)$ from the previous part, such that the real part of $h(z) = h_1(z)+h_2(z)$ satisfies the boundary condition everywhere on the cylinder, as well as the boundary condition at infinity. To do this assume $h_2(z) = az^n$, and find the values for constants a and n for which the boundary conditions are satisfied everywhere on the cylinder. (*Hint*: Use polar coordinates. The result, which you are supposed to derive, is $h(z) = v_0(z + R^2/z)$.)

Solution: The boundary condition at the cylinder is $\partial f/\partial r = 0$, where $f(\rho, \theta)$ is the velocity potential, i.e. the real part of $h(z)$. This real part is

$$f(z) = Re(h(z)) = Re(v_0 z + az^n) = Re(v_0 re^{i\theta} + ar^n e^{in\theta}) \quad (1)$$
$$= v_0 r \cos\theta + ar^n \cos n\theta, \quad (2)$$

so the radial derivative evaluated at $r = R$ is

$$\partial f/\partial r|_{r=R} = v_0 \cos\theta + naR^{n-1} \cos n\theta. \quad (3)$$

This must vanish for every θ around the cylinder, which is only possible if $n = 1$ or $n = -1$. The first possibility is no good, since it will require $a = -1$ so the potential f would vanish everywhere. Thus $n = -1$ and $a = v_0 R^2$. The velocity potential is thus the real part of

$$h(z) = v_0(z + R^2 z^{-1}). \quad (4)$$

(c) Using your result from the previous part, find the velocity (magnitude and direction) at the point $(x, y) = (0, R)$ on the surface of the cylinder. How does it compare with v_0?

Solution: By virtue of the boundary condition the flow is in the x-direction. The speed using problem **??** is $|dh/dz| = v_0|1 - R^2/z^2|$. At $(x, y) = (0, R)$ we have $z = iR$, so the speed is just $2v_0$, twice the asymptotic velocity.

Alternatively, the velocity at any point is $\nabla f = \nabla Re(h)$. We have

$$f(x, y) = v_0(x + R^2 x/(x^2 + y^2)). \tag{5}$$

Rather than writing out the gradient in all detail at a general value of (x, y) and then imposing $(x, y) = (0, R)$, let's be lazy and look ahead to what will vanish when we evaluate at that point. First, the derivative wrt y will of course vanish since we imposed that as a boundary condition the derivative perpendicular to the cylinder is zero at the cylinder. You can also easily see this by inspection, putting $x = 0$. In computing the derivative wrt x evaluated at $(0, R)$ we can ignore the derivative of the denominator $(x^2 + y^2)$ because the numerator x vanishes, so only the derivative of the numerator contributes:

$$\mathbf{v}(0, R) = v_0(1 + R^2/R^2)\hat{\mathbf{x}} = 2v_0\hat{\mathbf{x}}. \tag{6}$$

(d) Find the equation for the flow line that goes through the point $(x, y) = (0, y_0)$ (with $y_0 > R$). (The equation should involve x, y, y_0, v_0 and R.)

Solution: $Im(z^{-1}) = Im[(x - iy)/(x^2 + y^2)] = -y/(x^2 + y^2)$. Thus the flow lines are given by $Im(h) = v_0[y - R^2 y/(x^2 + y^2)] = const$. The constant is equal to $Im[h(0, y_0)] = v_0(y_0 - R^2/y_0)$.

(e) Find the asymptotic y value y_∞ when $x \to \infty$ for the flow line of problem 4d.

Solution: Setting $Im[h(0, y_0)] = Im[h(\infty, y_\infty)] = v_0(y_\infty)$ yields
$y_\infty = y_0 - R^2/y_0$.

(f) Sketch the flow lines of problem 4d for the cases $y_0 = R$ and $y_0 = 2R$, along with the circle of radius R representing the cross-section of the cylinder.

Solution: The $y_0 = R$ flowline coincides with the circular boundary $x^2+y^2 = R^2$ of the cylinder. The $y_0 = 2R$ flowline has $y_\infty = (3/2)R$, so bumps upward by $R/2$ as it goes past the cylinder.