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1. (a) Evaluate $\int z\,dz$ along the following two contours connecting -1 to 1: (i) from -1 to 1 along the real axis, and (ii) along the semicircle of radius 1 in the complex plane. (Do both contour integrals explicitly. Use the polar angle as the variable of integration for (ii), and use $x = Re(z)$ as the variable of integration for (i).) Explain how you could have known the two integrals would agree without even evaluating them. (b) Repeat part (a) for the integral $\int z^*\,dz$, and explain why the two contour integrals do *not* agree in this case. [7+3=10 pts.]

   Solution:
   (a)
   (i) Along contour $\mathcal{C}_i$, $z = x$, $dz = dx$, and $\int_{\mathcal{C}_i} z\,dz = \int_{-1}^{1} x\,dx = 0$.
   (ii) Along contour $\mathcal{C}_{ii}$, $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta$, and $\int_{\mathcal{C}_{ii}} z\,dz = i\int_{\pi}^{0} e^{i2\theta}\,d\theta = 0$. The function z is analytic everywhere, in particular in between the two contours. Hence the two integrals must agree.
   (b)
   (i) Along contour $\mathcal{C}_i$, $z^* = x$, $dz = dx$, and $\int_{\mathcal{C}_i} z^*\,dz = \int_{-1}^{1} x\,dx = 0$.
   (ii) Along contour $\mathcal{C}_{ii}$, $z^* = e^{-i\theta}$, $dz = ie^{i\theta}d\theta$, and $\int_{\mathcal{C}_{ii}} z\,dz = i\int_{\pi}^{0} d\theta = -i\pi$. The function z^* is *not* analytic, hence the two integrals need not agree.

2. **Fluid flow and analytic functions**: Problem 16.3 c,d,e,f [2+3+3+2=10 pts.]

   Solution:
   c) $\nabla \ln r = \hat{\mathbf{r}}/r$.

   d) In polar coordinates (see (10.35) for the Laplacian in cylindrical coordinates), $\nabla^2(\ln r) = r^{-1}\partial_r(r\partial_r \ln r) = r^{-1}\partial_r(1) = 0$. In Cartesian coordinates: $(\ln r)_{,xx} = (xr^{-2})_{,x} = r^{-2} - 2x^2r^{-4}$. Similarly with $x \leftrightarrow y$. Add them together to find $\nabla^2(\ln r) = 2r^{-2} - 2(x^2+y^2)r^{-4} = 0$.

   Since $\ln r$ diverges to negative infinity at $r = 0$, it is not clear from these calculations what the value of the Laplacian is there. In fact the divergence of the gradient cannot be zero there, since the flow is emerging from that point. Mathematically, the 2d form of the divergence theorem on a disk D centered on the origin yields $\int_D \nabla^2 \ln r\,dA = \oint_{\partial D}(\nabla \ln r)\cdot\hat{\mathbf{r}}\,r d\theta = 2\pi$. Evidently, then $\nabla^2 \ln r$ is infinitely positive at the point $r = 0$, in such a way that its integral is 2π. (That is, $\nabla^2 \ln r = 2\pi\delta^{(2)}(r)$, where $\delta^{(2)}(r)$ is the 2d Dirac delta function.)

   e) If $z = re^{i\varphi}$ then $Re(\ln z) = \ln r$ and $Im(\ln z) = \varphi + 2\pi n$ for integer n.

   f) $h = (\dot{V}/2\pi)\ln z$. Lines where $g = Im(h)$ are constant are lines of constant φ, i.e. radial lines, which are tangent to $\mathbf{v}$ everywhere, so they are the flow lines.

3. **Vortex**: The velocity potential for a point source of fluid flow is given by the real part of $h(z) = k \ln z$ (where k is a constant), as shown in the previous problem. Show that the *imaginary part* of $h(z)$ is the velocity potential for a point vortex. Do this by showing that the flow lines are circles centered on $z = 0$, and find the relation between k and the circulation. [5 pts.]

   The potential is $f = k\varphi$, so $\mathbf{v} = \nabla f = kr^{-1}\hat{\boldsymbol{\varphi}}$. (This can be obtained from the formula (5.48) for gradient in cylindrical coordinates. Or, just reason that the maximum rate of change of φ is in the $\hat{\boldsymbol{\varphi}}$ direction, and the rate of change with respect to *distance* in that direction is $d\theta/(rd\theta) = 1/r$.) Hence the flow lines are circles centered on the origin. The circulation is $\oint \mathbf{v} \cdot \mathbf{dl} = \int_0^{2\pi} kr^{-1} r d\varphi = 2\pi k$.

4. (a) Show that a flow with complex velocity potential $h(z)$ has speed $|dh/dz|$. (*Hint*: Use the fact that $dh/dz = \partial_x h$, and use the Cauchy-Riemann equations. (b) Apply this to the previous two problems to find the flow speed as a function of r. [8+2=10 pts.]

   Solution: (a) Let $h = f + ig$, where f and g are real. Then $dh/dz = \partial_x h = \partial_x f + i\partial_x g = \partial_x f - i\partial_y f = v_x - iv_y$, so $|dh/dz| = \sqrt{v_x^2 + v_y^2}$.
   (b) In both cases, $h(z) \propto \ln z$, so the speed is proportional to $|1/z| = 1/r$.

5. Consider potential flow in the wedge $0 < \theta < \alpha$ of the plane, bounded by walls at $\theta = 0$ (the x axis) and $\theta = \alpha$. At the walls the velocity must be parallel to the walls. (a) Show that the velocity potential $h(z) = Az^{\pi/\alpha}$ satisfies this boundary condition. (b) Sketch the flow lines for this potential for the cases $\alpha = \pi/2$, $\alpha = \pi$, $\alpha = 3\pi/2$, and $\alpha = 2\pi$. (c) Find the speed of the flow as a function of position on the plane for general α. Explain why your result makes sense for the case $\alpha = \pi$. (d) If $\alpha < \pi$, the speed goes to infinity as r goes to infinity. Explain how this is compatible with incompressibility of the flow. [3+3+2+2=10 pts.]

   Note that if $\alpha < \pi$ the speed goes to zero at the vertex of the wedge, whereas if $\alpha > \pi$ the speed goes to infinity at the vertex. I read that this is why the wind whistles when going over a pointed obstacle: locally supersonic velocities are reached.

   Solution: (a) To check the boundary condition, we could evaluate the gradient of the real part and verify that it is parallel to the boundary. However, it is even simpler just verify that the flow lines coincide with the boundary, that is, that the imaginary part of $h(z)$ is constant on the boundaries. We have $Im(z^{\pi/\alpha}) = Im(re^{i\pi\theta/\alpha})$. At $\theta = 0$ and $\theta = \alpha$ this is equal to zero, so is indeed constant.

   (b) For $\alpha = \pi/2$ (right angle wedge), $z^{\pi/\alpha} = z^2 = (x^2 - y^2) + i2xy$, where $z = x + iy$. The flow lines are thus given by $xy = constant$. These are hyperbolae, asymptotic to the x and y axes. For $\alpha = \pi$ (half-plane), $z^{\pi/\alpha} = z = x + iy$. The flow lines are $y = constant$ lines. For $\alpha = 3\pi/2$ (right angle 'shelf'), $z^{\pi/\alpha} = z^{2/3} = r^{2/3}e^{i2\theta/3}$. The flow lines are $r^{2/3}\sin(2\theta/3) = constant$ lines. These are harder to describe in detail, but they are tangent to the shelf boundary, and wrap around it. For $\alpha = 2\pi$ (spike formed by positive real axis), $z^{\pi/\alpha} = z^{1/2}$. The flow lines are $r^{1/2}\sin(\theta/2) = constant$ lines. These are harder to describe in detail, but they are tangent to the spike on both sides, and wrap around it.

(c) The speed is $|dh/dz| = A(\pi/\alpha)|z^{\pi/\alpha-1}| = A(\pi/\alpha)r^{\pi/\alpha-1}$. For $\alpha = \pi$ this is just the constant A, which makes sense since the flow is just uniform flow in the x direction.

(d) The flow is asymptotic to the edges of the wedge. Since the flow is incompressible, it must be flowing infinitely fast where the flow lines are concentrated infinitely closely to the boundary.

6. Consider the real integral

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$$

where a and b are positive real numbers.

(a) Evaluate the integral using contour integration assuming $a \neq b$ (so there are only simple poles).

(b) Evaluate the integral using contour integration assuming $a = b$ from the beginning (so the poles are of order 2), and then check that you recover the same result by setting $a = b$ in the result of the previous part.

Hint: You can check your result by testing for a few properties: the integral is manifestly positive and symmetric under interchange of a and b, and scales as λ^{-3} under the scaling $a \to \lambda a$ and $b \to \lambda b$. (Equivalently, if you think of a and b as having dimensions, the integral has the same dimensions as a^{-3}.) [5+5=10 pts.]

Solution: We can think of it as a complex integral over a contour running along the real axis. We can close the contour at infinity since the integrand goes like $\sim 1/R^2$ on an arc of radius R, and the arc length goes as $\sim R$, so the integral over an arc at radius R goes like $\sim R/R^2 = 1/R$, which vanishes as $R \to \infty$. The integrand $(z^2+a^2)^{-1}(z^2+b^2)^{-1}$ has simple poles at $z = \pm ia, \pm ib$. If we close the contour in the upper half-plane, only the poles at ia and ib contribute. The integrand has the form $f(z)/g(z)$, where $f(z)$ is nonzero at the simple poles of $1/g(z)$, hence the residue at the simple pole at z_0 is $f(z_0)/g'(z_0)$. The residue at the simple pole at ia is therefore $1/2ia(b^2-a^2)$. The integrand is symmetric under $a \leftrightarrow b$, so the residue at ib is $1/2ib(a^2-b^2)$. The value of our original integral is $2\pi i$ times the sum of these residues, $I = \pi/ab(a+b)$ (after a little algebra). This is indeed positive, symmetric in a and b, and has the same dimensions as a^{-3}.

If $a = b$ from the beginning, there are only poles of order two at $\pm ia$. The integrand can be written as $(z+ia)^{-2}(z-ia)^{-2}$, which has the form $f(z)/(z-z_0)^2$, so the residue is $f'(z_0)$. At ia this yields $-i/4a^3$, so the integral is $(2\pi i)(-i/4a^3) = \pi/2a^3$. This agrees with the previous result when $b \to a$.